Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 3, No. 1–2, Series A

Generalized Whitehead Products and Homotopy Groups of Spheres

By Hirosi TODA

(Received September 15, 1952)

Introduction

A fundamental problem in algebraic topology, the calculation of homotopy groups $\pi_r(S^n)$ of spheres, was initiated by studies of several authors; Brouwer's degree, Hopf's invariant and Freudenthal's suspension method. Recently, G. W. Whitehead [22] [23][1] generalized Hopf's invariant and Freudenthal's invariant to enumerate several non-trivial homotopy groups of spheres. It is reported that H. Cartan, P. Serre, G. W. Whitehead, and W. S. Massey[2] have obtained a number of remarkable results[2], applying Eilenberg-MacLane's cohomology theory of a group complex.

Methods employed here by author, are rather intuitive. Making use of recent results due to S. Eilenberg and S. MacLane [7], he constructs an elementary CW-complex K_n, the n-section K_n^n of which is an n-sphere S^n, such that excepting $\pi_n(K_n)=Z$, all the other homotopy groups vanish. Generators in $\pi_r(S^n)$, which are essential in the $(n+k-1)$-skelton K_n^{n+k-1} and inessential in K_n^{n+k}, can be represented by the image of the boundary homomorphism: $\pi_{r+1}(K_n^{n+k}, K_n^{n+k-1}) \to \pi_r(K_n^{n+k-1})$. Thus, generators of $\pi_r(S^n)$ can be realized by adequately chosen maps in virtue of the construction of the complex K_n. Main results in this paper are stated as follows.

Theorem i) $\pi_{n+3}(S^n)=Z_{24}$ *for* $n\geqq 5$, *the genrator of which is represented by* $(n-4)$*-fold suspension of the Hopf's fibre map*: $S^7\to S^4$.

ii) $\pi_{n+4}(S^n)=0$ *for* $n\geqq 6$,

iii) $\pi_{n+5}(S^n)=0$ *for* $n\geqq 7$,

iv) $\pi_{n+6}(S^n)=Z_2$ *or* Z_2+Z_2 *for* $n\geqq 8$,

v) $\pi_{n+7}(S^n)$ *is the direct sum of* Z_{15} *and a group of order* $2^k (3\leqq k\leqq 8)$ *for* $n\geqq 9$.

vi) $\pi_{n+8}(S^n)$ *is a group of order* 2^k *for* $n\geqq 10$.

In Chapter 1 various kinds of notations are given and the excision theorem[3] due to A. L. Blakers and W. S. Massey is stated in order to be available under the removal of the restriction in dimensions. In Chapter 2 Whitehead product

1) Numbers in blackets refer to the references cited at the end of the paper.
2) *Cf.* [4], [14], [15], [16] and Bull. Amer. Math. Soc. U.S.A. 57 (1951) abstruct 544.
3) Theorem I of [3].

is generalized to get certain types of products, called generalized Whitehead product[4], which have much to do with the Hopf construction of G. W. Whitehead. In Chapter 3 generalized Hopf invariant and Freudenthal invariant are systematically discussed as a Hopf homomorphism of a triad $\pi_n(S^r; E^r_+, E^r_-)$. Generalizing this homomorphism to define a Hopf homomorphism of $\pi_n(X^*; \mathcal{E}^r, X)$, we obtain that $\pi_n(X^*; \mathcal{E}^r, X)$ has a direct factor isomorphic to $\pi_{n-r+1}(X, \dot{\mathcal{E}}^r)\otimes\pi_r(\mathcal{E}^r, \dot{\mathcal{E}}^r)$ in lower dimensional cases. In Chapter 4, essential elements in homotopy groups $\pi_n(S^r)$ of spheres of special dimensions, are given and also their essentiality is shown by means of Hopf invariant. In Chapter 5, a homomorphism $T: {}_2[\pi_n(X)]\to\pi_{n+1}(X)/2\pi_{n+1}(X)$ is introduced in order to consider the element of order four in $\pi_{n+3}(S^n)(n\geqq 3)$, which Barratt and Paecher obtained recently. In Chapter 6 it is shown how the suspension homomorphism of Eilenberg-MacLane[5] is interpreted as homomorphisms of homology groups of K_n, K_{n+1} by making use of their recent results. Chapter 7–8 involve our principal results. Making use of preparations in the previous chapters, we can compute automatically homotopy groups $\pi_n(S^r)$ of spheres. We calculate homotopy groups of the n-fold suspended space Y^{n+1} of the projective plane, making use of T-homomorphism in Chapter 5.

Chapter 1. Preliminaries

i) In this section, we shall describe several notations, which will be used throughout this paper.

Symbols $(X; X_1, \dots, X_n, X_0)$, $(X; X_1, \dots, X_n)$, (X, A, x_0), (X, A) and (X, x_0) indicate systems of topological spaces such that $X\supset X_i$, $X_1\cap\cdots\cap X_n\ni x_0$, $X\supset A\ni x_0$ and $X\ni x_0$. A mapping $f: X\to X'$ is a continuous function of X to X', and if $f(X_i)\supset X_i'$ and $f(x_0)=x_0'$, the mapping f is indicated by $f: (X; X_1, \dots, X_n, x_0)\to(X'; X_1', \dots, X_n', x_0')$. A homotopy f_t[6] $: (X; X_1, \dots, X_n, x_0)\to(X'; X_1', \dots, x_n', x_0')$ means that the homotopy $f_t: X\to X'$ carias the subsets X_i and x_0 to X_i' and x_0' respectively. If $f: X\to Y$ and $g: Y\to Z$ are mappings, a composite map $g\circ f: X\to Z$ is given by $(g\circ f)(x)=g(f(x))$ for $x\in X$.

$x=(x_1, \dots, x_n)$ indicates a point of the real Cartesian space C of infinite dimension having the i-th coordinate x_n for $i\leqq n$ and 0 for $i>n$, thus $(x_1, \dots, x_n)$ and $(x_1, \dots, x_n, 0, \dots, 0)$ indicate the same point of C.[7] Define subspaces of C by

4) Products of this sort are also provided by A. L. Blakers and W. S. Massey; cf. Bull. Amer. Math. Soc. U. S. A. 57 (1951) abstruct 165.
5) *Cf.* [7].
6) The homotopy is indicated by symbol: $f_0\simeq f_1$.
7) The n-dimensional cartesian space is denoted by C^n.

$$E^n=\{(x_1,\dots,x_n)|\sum x_i^2\leqq 1\},\quad S^n=\{(x_1,\dots,x_{n+1})|\sum x_i^2=1\},$$
$$E^n_{+i}=\{(x_1,\dots,x_{n+1})\in S^n|x_i\geqq 0\},\quad E^n_+=E^n_{+(n+1)},$$
$$E^n_{-i}=\{(x_1,\dots,x_{n+1})\in S^n|x_i\leqq 0\},\quad E^n_-=E^n_{-(n+1)},$$
$$S^n_i=\{(x_1,\dots,x_{n+1})\in S^n|x_i=0\},\quad y_*=(1,0,\dots,0):$$
(1.1) $$I^n=\{(x_1,\dots,x_n)|0\leqq x_i\leqq 1\},\quad \dot{I}^n=\{(x_1,\dots,x_n)\in I^n|\Pi x_i(1-x_i)=0\},$$
$$I^n_+=\{(x_1,\dots,x_n)\in I^n|x_n\geqq\tfrac{1}{2}\},\qquad I^n_-=\{(x_1,\dots,x_n)\in I^n|x_n\leqq\tfrac{1}{2}\},$$
$$I^n_i=\{(x_1,\dots,x_{n+1})\in I^{n+1}|x_i=0\},\quad J^n_i=Cl(\dot{I}^{n+1}-I^n_i),$$
$$\dot{I}^n_i=J^n_i\cap I^n_i,\quad J^n=J^n_{n+1},\quad J^n_0=Cl(\dot{I}^{n+1}_n-I^{n-1}),$$
$$K^n_{ij}=J^n_i\cap J^n_j,\quad K^n=K^n_{n,n+1}\ \text{and}\ 0_*=(0,\dots,0).$$

Thus $E^{n+1}\supset E^n\supset S^{n-1}\supset E^{n-1}_{+i}\supset S^{n-1}_i$, $E^{n+1}-\text{Int.}E^{n+1}=S^n=E^n_{+i}\cup E^n_{-i}$, $E^n_{+i}\cap E^n_i=S^{n-1}_i$, $I^n\supset\dot{I}^n\supset J^{n-1}_i\supset K^{n-1}_{ij}\ni 0_*$ and I^n—Int. $I^n=\dot{I}^n=J^{n-1}_i\cup I^{n-1}_i=K^{n-1}_{ij}\cup I^{n-1}_i\cup I^{n-1}_j$.

Let

(1.2) $$P_n:(J^n,\dot{I}^n,0_*)\to(I^n,\dot{I}^n,0_*)$$
and $$P_n':(K^n;J^{n-1},J^{n-1}_0,0_*)\to(I^n;J^{n-1},I^{n-1},0_*)$$

be projections from the points $(1/2,\dots,1/2,-1)\in C^{n+1}$ and $(1/2,\dots,1/2,0,-1)\in C^{n+1}$ respectively, then P_n and P_n' are homeomorphisms.

Let $\rho_n(\theta):(E^{n+1},S^n)\to(E^{n+1},S^n)$ be the rotation through θ given by

(1.3) $$\rho_n(\theta)(x_1,\dots,x_{n+1})=(x_1,\dots,x_{n-1},\cos\theta\cdot x_n-\sin\theta\cdot x_{n+1},\sin\theta\cdot x_n+\cos\theta\cdot x_{n+1})$$

Define a mapping $d_n:(S^n\times E^1,S^n\times S^0\cup y_*\times E^1)\to(S^{n+1},y_*)$ by

(1.4)
$$\begin{aligned} d_n(x,t)&=(t+(1-t)x_1,(1-t)x_2,\dots,(1-t)x_{n+1},(2t(1-t)(1-x_1))^{\frac{1}{2}}) && 0\leqq t\leqq 1,\\ &=(-t+(1+t)x_1,(1+t)x_2,\dots,(1+t)x_{n+1},-(2-t(1+t)(1-x_1))^{\frac{1}{2}}) && -1\leqq t\leqq 0,\end{aligned}$$

then d_n maps $S^n\times E^1-(S^n\times S^0\cup y_*\times E^1)$ homeomorphically onto $S^{n+1}-y_*$, and

(1.4)′ $$d_n(E_{\pm i}\times E^1)=E^{n+1}_{\pm i}\ \text{for}\ 1<i\leqq n+1.$$

Define a mapping $\psi_n:(I^n,\dot{I}^n)\to(S^n,y_*)$ inductively by setting

(1.5) $$\psi_1(x_1)=\rho_1(2\pi x_1)(y_*)$$
and $$\psi_n(x_1,\dots,x_n)=d_{n-1}(\psi_{n-1}(x_1,\dots,x_{n-1}),2x_n-1)\ \text{for}\ n\geqq 2,$$

then ψ_n maps Int. I^n homeomorphcally onto S^n-y_*.

Let $\varepsilon_1:(\dot{I}^2,0_*)\to(S^1,y_*)$ be a homeomorphism given by $\varepsilon_1(x_1,0)=\rho(\pi x_1)(y_*)$, and for $x\in J^1$ by $\varepsilon_1(x)=\rho_1(\pi)\bar{\varepsilon}_1P_1(x)$.

Define homeomorphisms $\varepsilon_n:(\dot{I}^{n+1},0_*)\to(S^n,y_*)$ and $\bar{\varepsilon}_n:(I^n,\dot{I}^n)\to(E^n,S^{n-1})$ inductively by setting $\bar{\varepsilon}_n\left(\frac{1+t(2x_1-1)}{2},\dots,\frac{1+t(2x_n-1)}{2}\right)=(tx_1',\dots,tx_n')$ for

$(x_1', \dots, x_n')=\varepsilon_n(x_1, \dots, x_n)$ and $t\in I^1$, and by setting $\varepsilon_n(x)=p_-^{-1}(\bar\varepsilon_n(x))$ and $\varepsilon_n(P_n(x))=p_+^{-1}(\bar\varepsilon_n(x))$ for $x\in I^n$, where $p_+: E_+^n\to E^n$ and $p_-: E_-^n\to E^n$ are projections given by $p_+(x_1, \dots, x_{n+1})=(x_1, \dots, x_n, 0)$. Note that (for $n>1$)

$$(1.6)\qquad \varepsilon_n(I^n)=E_-^n,\quad \varepsilon_n(J^n)=E_+^n \text{ and } \varepsilon_n|\dot I^n=\varepsilon_{n-1}.$$

The calles and their boundaries $I^n, \dot I^n, J^n, E^n, S^{n-1}, E_+^n$, and E_-^n are orientable such that ψ_n, ε_n and p_- preserve the orientations and P_n and p_+ reverse the orientations.

Denote a subspace $S^n\times y_*\cup y_*\times S^q$ of $S^p\times S^q$ by $S^p\vee S^q$, and define a mapping $\varphi_n: (S^n; E_+^n, E_-^n, y_*)\to(S^n\vee S^n; S^n\times y_*, y_*\times S^n, y_*\times y_*)$ by $(x\in S^{n-1}, t\in E^1)$

$$(1.7)\qquad \begin{aligned}\varphi_n(d_{n-1}(x, t)) &= (\rho_n(\pi/2)\circ d_{n-1}(x, 2t-1), y_*) && 0\leqq t\leqq 1,\\ &= (y_*, \rho_n(-\pi/2)\circ d_{n-1}(x, 2t+1)) && -1\leqq t\leqq 0.\end{aligned}$$

φ_n maps Int. E_+^n and Int. E_-^n homeomorphically onto $(S^n-y_*)\times y_*$ and $y_*\times(S^n-y_*)$ preserving orientations, and $\varphi_n(E_{+n}^n)=E_+^r\vee E_-^r$ and $\varphi_n(E_{-n}^n)=E_-^r\vee E_+^r$. Let $\sigma_n: (S^n\times S^n, S^n\vee S^n)\to(S^n\times S^n, S^n\vee S^n)$ be a homeomorphism given by

$$(1.8)\qquad \sigma_n(x, y)=(y, x),\qquad x, y\in S^n,$$

then we have

$$(1.9)\qquad \sigma_n\circ\varphi_n=\varphi_n\circ\rho_n(\pi).$$

Define a mapping $\psi_{p,q}: (I^{p+q}, \dot I^{p+q})\to(S^p\times S^q, S^p\vee S^q)$ by

$$(1.10)\qquad \psi_{p,q}(x_1, \dots, x_{p+q})=(\psi_p(x_1, \dots, x_p), \psi_q(x_{p+1}, \dots, x_{p+q})).$$

$\psi_{p,q}$ maps Int. $I^{p+q}=I^{p+q}-\dot I^{p+q}$ homeomorphically onto $S^p\times S^q-S^p\vee S^q$, hence there is unique mapping $\phi_{p,q}: (S^p\times S^q, S^p\vee S^q)\to(S^{p+q}, y_*)$ such that

$$(1.11)\qquad \phi_{p,q}\circ\psi_{p,q}=\psi_{p+q}.$$

Define a mapping $\Phi_{p,q}: \dot I^{p+1}\times\dot I^{q+1}\times E^1\to\dot I^{p+q+2}$ by

$$(1.12)\qquad \begin{aligned}\Phi_{p,q}(x, y, t) &= ((1-t)x_1, \dots, (1-t)x_{p+1}, y_1, \dots, y_{q+1}) && 0\leqq t\leqq 1,\\ &= (x_1, \dots, x_{p+1}, (1+t)y_1, \dots, (1+t)y_{q+1}) && -1\leqq t\leqq 0,\end{aligned}$$

where $x=(x_1, \dots, x_{p+1})\in\dot I^{p+1}$, $y=(y_1, \dots, y_{q+1})\in\dot I^{q+1}$ and $t\in E^1$, then $\phi_{p,q}|\dot I^{p+1}\times\dot I^{q+1}\times$ Int. E^1 is a homeomorphism.

ii) homotopy groups

Define *a sum* $f+_i g: (I^n, \dot I^n)\to(X, x_0)$ *of* f *and* $g: (I^n, \dot I^n)(X, x_0)$ *on the* x_i*-axis* $(1\leqq i\leqq n)$ by

$$(1.13)_1\qquad \begin{aligned}(f+_i g)(x_1, \dots, x_n) &= f(x_1, \dots, x_{i-1}, 2x_i, x_{i+1}, \dots, x_n) && 0\leqq x_i\leqq 1,\\ &= g(x_1, \dots, x_{i-1}, 2x_i-1, x_{i+1}, \dots, x_n) && -1\leqq x_i\leqq 0,\end{aligned}$$

and also define *an inverse* $-_i f: (I^n, \dot I^n)\to(X, x_0)$ of f by

$$(1.13)_2 \qquad (-_i f)(x_1, \dots, x_n) = f(x_1, \dots, x_{i-1}, 1-x_i, x_{i+1}, \dots, x_n).$$

It is easily seen that the sums $f +_i g$ on different two axes are homotopic to each other, and the homotopy classes of f form the (*absolute*) *homotopy group* $\pi_n(X, x_0)$ with respect to to the above addition.

A mapping $f: (S^n, y_*) \to (X, x_0)$ is called a *representative* of $\alpha \in \pi_n(X, x_0)$ if the homotopy class of the composite map $f \circ \psi_n: (I^n, \dot{I}^n) \to (X, x_0)$ is α. Define *a sum* $f+g$ of f and $g: (S^n, y_*) \to (X, x_0)$ and *an inverse* $-f$ of f by

$$(1.14) \qquad \begin{aligned} (f+g)(d_{n-1}(x, t)) &= f(d_{n-1}(x, 2t+1)) \quad -1 \leqq t \leqq 0, \\ &= g(d_{n-1}(x, 2t-1)) \quad 0 \leqq t \leqq 1, \end{aligned}$$

and
$$(-f)(d_{n-1}(x, t)) = f(d_{n-1}(x, -t)),$$

then $\psi_n(f+g) = \psi_n(f) +_n \psi_n(g)$, $\psi_n(-f) = -_n\psi_n(f)$ and $\psi_n(f) \simeq \psi_n(g)$ implies $f \simeq g$. Therefore $\pi_n(X, x_0)$ may be regarded as the set of the homotopy classes of $f: (S^n, y_*) \to (X, x_0)$ with addition in (1.14).

A mapping $f: (\dot{I}^{n+1}, 0_*) \to (X, x)$ is called a *representative* of $\alpha \in \pi_n(X, x_0)$ if there is a mapping $f': (\dot{I}^{n+1}, 0) \to (X, x_0)$ such that $f \simeq f'$, $f'(J^n) = x_0$ and the class of $f'|I^n: (I^n, \dot{I}^n) \to (X, x_0)$ is α. It is not so difficult to show that

(1.15) *If* $f: (S^n, y_*) \to (X, x_0)$ *is a representative of* α, *then the composite map* $f \circ \varepsilon_n: (\dot{I}^{n+1}, 0_*) \to (X, x_0)$ *is also a representative of* α.

The relative homotopy group $\pi_n(X, A, x_0)$ is a set of homotopy classes of mappings: $(I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$ with addition which is represented by a sum and an inverse on the x_i-axis $(1 \leqq i \leqq n-1)$ as in $(1.13)_1$ and $(1.13)_2$. A mapping $f: (I^n, \dot{I}^n, 0_*) \to (X, A, x_0)$ is called a representative of $\alpha \in \pi_n(X, A, x_0)$ if there is a mapping f'[8]$: (I^n, \dot{I}^n, 0_*) \to (X, A, x_0)$ such that $f \simeq f'$, $f'(J^n) = x_0$ and the class of $f': (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$ is α. Also a mapping $f: (E^n, S^{n-1}, y_*) \to (X, A, x_0)$ is called a representative of $\alpha \in \pi_n(X, A)$, if the composite map $f \circ \bar{\varepsilon}_n: (I^n, \dot{I}^n, 0_*) \to (X, A, x_0)$ is a representative of α.

The triad homotopy group $\pi_n(X; A, B, x_0)$ is a set of homotopy classes of mappings: $(I^n; I_+^{n-1}, I_-^{n-1}, J^{n-1}) \to (X; A, B, x_0)$ with addition which is represented by a sum and an inverse on the x_i-axis $(1 \leqq i \leqq n-2)$ as in (1.13). Since $\psi_{n-1}: (I^{n-1}, \dot{I}^{n-1}) \to (S^{n-1}, y_*)$ maps I_+^{n-1} and I_-^{n-1} to E_+^{n-1} and E_-^{n-1} respectively, there is a mapping $\bar{\psi}_n: (I^n; I_+^{n-1}, I_-^{n-1}, J^n) \to (E^n; E_+^n, E_-^n, y^*)$ such that $\bar{\psi}_n | I^{n-1} = \psi_{n-1}$ and $\bar{\psi}_n$ maps $I^n - J^{n-1}$ homeomorphically onto $E^n - y_*$. As is easily seen, any extensions $\bar{\psi}_n'$ of $\psi_{n-1} = \bar{\psi}_n' | I^{n-1}$ are homotopic to each other. A mapping $f: (E^n; E_+^{n-1}, E_-^{n-1}, y_*) \to (X; A, B, x_0)$ is called a representative of $\alpha \in \pi_n(X; A, B, x_0)$ if the composite map $f \circ \bar{\psi}_n: (I^n; I_+^{n-1}, I_-^{n-}, J^{n-1}) \to (XA, B, x_0)$ represents α. Also a mapping $f: (I^n; J^{n-1}, I^{n-1}, 0_*) \to (X; A, B, x_0)$ is called a

8) The existence of such mapping is clear.

representative of α, if the composite map $f\circ\bar{\varepsilon}_n^{-1}:(E^n;E_+^{n-1},E_-^{n-1},y_*)\to(X;A,B\ x_0)$ is a representative of α.

Let $f:(X,x_0)\to(Y,y_0)$ be a mapping, for any mappings g_1 and $g_2:(I^n,\dot{I}^n)\to(X,x_0)$ we have that $g_1\simeq g_2$ implies $f\circ g_1\simeq f\circ g_2$ and that $f\circ(g_1+_ig_2)=(f\circ g_1)+_i(f\circ g_2)$. Therefore f induces a homomorphism

(1.16) $$f^*;\ \pi_n(X,x_0)\to\pi_n(Y,y_0).$$

Similarly mappings $f_1:(X,A,x_0)\to(Y,B,y_0)$ and $f_2:(X;A,B,x_0)\to Y;C,D,y_0)$ induce homomorphisms

(1.16)′ $$f_1^*:\ \pi_n(X,A,x_0)\to\pi_n(Y,B,y_0),$$

and $$f_2^*:\ \pi_n(X;A,B,x_0)\to\pi_n(Y;C,D,y_0).$$

The mapping $f:(I^n,\dot{I}^n)\to(X,x_0)$ is regarded as the mapping $f:(I^n,I^{n-1},J^{n-1})\to(X,x_0,x_0)$ and this implies the natural isomorphism

(1.17) $$j';\ \pi_n(X,x_0)\to\pi_n(X,x_0,x_0).$$

The mapping $f:(I^n,I^{n-1},J^{n-1})\to(X,B,x_0)$ is regarded as the mapping $f:(I^n;J^{n-1},I^{n-1},K^{n-1})\to(X;x_0,B,x_0)$ and this implies the natural isomorphism

(1.17)′ $$j':\ \pi_n(X,B,x_0)\to\pi_n(X;x_0,B,x_0).$$

Define a boundary $\partial f:(I^{n-1},\dot{I}^{n-1})\to(A,x_0)$ of $f:(I^n,I^{n-1},J^{n-1})\to(X,A,x_0)$ by $\partial f=f\,|\,I^{n-1}$, then $f\simeq g$ implies $\partial f\simeq\partial g$ and $\partial(f+_ig)=\partial f+_i\partial f$ for $1\leqq i\leqq n-1$. Therefore we obtain *the boundary homomorphism*

(1.18) $$\partial;\ \pi_n(X,A,x_0)\to\pi_{n-1}(A,x_0)\quad\text{for } n\geqq 2$$

Define a boundary $\beta_+f:(I^{n-1};I^{n-2},J^{n-2})\to(A,A\frown B,x_0)$ of $f:(I^n;I_+^{n-1},I_-^{n-1},J^{n-1})\to(X;A,B,x_0)$ by $\beta_+f(x_1,\dots,x_n)=f(x_1,\dots,x_{n-2},2x_{n-1}-1,0)$ then $f\simeq g$ implies $\beta_+f\simeq\beta_+g$ and $\beta_+(f+_ig)=\beta_+f+_i\beta_+g$ for $1\leqq i\leqq n-2$. Therefore we obtain *the boundary homomorphism*

(1.18)′ $$\beta_+;\ \pi_n(X;A,B,x_0)\to\pi_{n-1}(A,A\frown B,x_0)\ \text{for } n\geqq 3.$$

The following properties are well known,

(1.19) i) *If f is the identity map, then f^* is the identity homomorphism.*

ii) $(f\circ g)^*=f^*\circ g^*$.

iii) *$f\simeq g$ implies $f^*=g^*$.*

(1.20) The sequence of the homomorphisms

$$\cdots\longrightarrow\pi_n(X,x_0)\xrightarrow{j}\pi_n(X,A,x_0)\xrightarrow{\partial}\pi_{n-1}(A,x_0)\xrightarrow{i^*}\pi_{n-1}(X,x_0)\longrightarrow\cdots\quad(n>1)$$

is *exact*, where $i:A\to X$ is the injection and j^* is the composite homomorphism $\pi_n(X,x_0)\xrightarrow{j'}\pi_n(X,x_0,x_0)\xrightarrow{\text{injection}}\pi_n(X,A,x_0)$. And also the sequence of the homomorphisms

$$\cdots\to\pi_n(X,B,x_0)\xrightarrow{j'}\pi_n(X;A,B,x_0)\xrightarrow{\beta_+}\pi_{n-1}(A,A\cap B,x_0)\xrightarrow{i^*}\pi_{n-1}(X,B,x_0)\to\cdots \quad (n>2)$$

is *exact*, where $i;(A,A\cap B,x_0)\to(X,B,x_0)$ is the injection and j^* is the composite homomorphism $\pi_n(X,B,x_0)\xrightarrow{j'}\pi_n(X;x_0,B,x_0)\to\pi_n(X;A,B,x_0)$.

(1.21) In the following diagrams the commutativity relations hold;

$$\begin{array}{ccccccccc}\cdots\to&\pi_n(X,x_0)&\to&\pi_n(X,A,x_0)&\to&\pi_{n-1}(A,x_0)&\to&\pi_{n-1}(X,x_0)&\to\cdots\\ &\downarrow f^*&&\downarrow f^*&&\downarrow (f\mid A)^*&&\downarrow f^*&\\ \cdots\to&\pi_n(Y,y_0)&\to&\pi_n(Y,B,y_0)&\to&\pi_{n-1}(B,y_0)&\to&\pi_{n-1}(X,y_0)&\to\cdots\end{array}$$

and

$$\begin{array}{ccccccccc}\cdots\to&\pi_n(X,B,x_0)&\to&\pi_n(X;A,B,x_0)&\to&\pi_{n-1}(A,A\cap B,x_0)&\to&\pi_{n-1}(X,B,x_0)&\cdots\\ &\downarrow g^*&&\downarrow g^*&&\downarrow (g\mid A)^*&&\downarrow g^*&\\ \cdots\to&\pi_n(Y,D,y_0)&\to&\pi_n(Y;C,D,y_0)&\to&\pi_{n-1}(C,C\cap D,y_0)&\to&\pi_{n-1}(Y,D,y_0)&\cdots\end{array}$$

where $f:(X,A,x_0)\to(Y,B,y_0)$ and $g;(X;A,B,x_0)\to(Y;C,D,y_0)$ are mappings.

Definition (1.22) $\pi_0(X)=0$ if and only if X is arcwise connected, $\pi_1(X,A,x_0)=0$ if and only if the injection homomorphism $i^*:\pi_1(A,x_0)\to\pi_1(X,x_0)$ is onto, and $\pi_2(X;A,B,x_0)=0$ if and only if the injection homomorphism $i^*:\pi_2(A,A\cap B,x_0)\to\pi_2(X,B,x_0)$ is onto.

X is called *n-connected* if $\pi_i(X,x_0)=0$ for $0\leq i\leq n$. (X,A,x_0) is called *n-connected* if $\pi_0(A,x_0)=\pi_0(X,x_0)=0$ and $\pi_i(X,A,x_0)=0$ for $1\leq i\leq n$. $(X;A,B,x_0)$ is called *n-connected* if $(A,A\cap B,x_0)$ and $(B,A\cap B,x_0)$ are 1-connected and $\pi_i(X;A,B,x_0)=0$ for $2\leq i\leq n$.

iii) The main theorem of Blakers and Massey [3] is described without restriction in lower dimension;

Theorem (1.23) *If $X=(Int\ A)\cup(Int\ B)$, $(A,A\cap B)$ is m-connected and $(B,A\cap B)$ is n-connected, then the triad $(X;A,B)$ is $(m+n)$-connected $(m\geq n\geq 1)$.*

For the case $n\geq 2$, the proof of theorem was given in [3].

We shall prove this theorem for the case $n=1$. With normalization process in §3 of [3], any elements of $\pi_q(X;A,B)$ is represented by normal form $f:(I^q;I^{q-1},J^{q-1})\to(X;A,B)$ such that

$$f^{-1}(A)\supset\bar{N}(M)\cup I^{q-1}\quad\text{and}\quad f^{-1}(B)\supset Cl(I^q-\bar{N}(M)).$$

Suppose $2\leq q\leq m+1$, then dim. $M\leq q-m-1\leq 0$, and therefore $\bar{N}(M)=\bigcup_i\sigma_i^q+\bigcup_j\tau_j^q$, where σ_i^q and τ_j^q are finite number of disjoint rectilinear closed cells in I^q-J^{q-1} such that $\sigma_i^q\cap I^{q-1}=\phi$ and $\tau_j^q\cap I^{q-1}$ are faces of τ_j^q.

Since $q\geq 2$, we can take segments t_j from each point of τ_j^q to point of I^{q-1} in $I^q-\dot{I}^q-\bar{N}(M)$, such that t_i are disjoint to each other. Set $P=\bigcup_j t_j\cup\bar{N}(M)\cup I^{q-1}$ and $Q=Cl(I^q-P^r)$, then interior of Q is an open q-cell and its boundary is $(P\cap Q)\cup J^{q-1}$ and $(P\cap Q)\cap J^{q-1}=\dot{I}^{q-1}$. Therefore $P\cap Q$ is a retract of Q and

let its retraction be $r_t: Q\to P_\cap Q$.
Since $(B, A_\cap B)$ is 1-connected, $f\,|\,\underset{j}{\cup} t_j$ is deformable into $A_\cap B$ relative to $\underset{j}{\cup} t_j$, and extendable to whole homotopy of $(I^q;I^{q-1},J^{q-1})$ such that the resulted map $f_1:(I^q;I^{q-1},J^{q-1})\to(X;A,B)$ satisfies the conditions

$$f_1^{-1}(A)\supset P \quad\text{and}\quad f_1^{-1}(B)\supset Q.$$

Equations $f_{1+t}|Q=f_1\circ r_t$ and $f_{1+t}|P=f_1|P$ define a homotopy of $f_1\simeq f_2$, and f_2 maps I^q in A. Also a retraction of I^q leads a null-homotopy of f_2 and nullhomotopy of f. Consequently any element of $\pi_q(X;A,B)$ is trivial for $2\leqq q\leqq m+1$, and the proof of theorem is comleted.

Let $\overline{A_\cap B}$, $\bar{A}$, $\bar{B}$ and $\bar{X}$ be subspaces of $X\times I^1$ given by $\overline{A_\cap B}=(A_\cap B)\times I^1$, $\bar{A}=A\times(0)^\cup\overline{A_\cap B}$, $\bar{B}=B\times(1)^\cup\overline{A_\cap B}$ and $\bar{X}=\bar{A}^\cup\bar{B}$, and let $\phi':(\bar{A},(A_\cap B)\times(1))\to(\hat{A},x_0)$ and $\phi:(A,A_\cap B)\to(\tilde{A},x_0)$ be mappings identifying the subsets $(A_\cap B)\times(1)$ and $A_\cap B$ to single points.

For convenience we shall give a sufficient condition to omit the condition $X=\text{Int.}\,A^\cup\text{Int.}\,B$ of (1.23).

Definition (1.24) The pair $(A, A_\cap B)$ is *smooth* if and only if there is a homotopy $h_t:(A,A_\cap B)\to(\bar{A},\overline{A_\cap B})$ such that $h_t(x)=(x,t)$ for $x\in A_\cap B$.

Lemma (1.25) i) *If $(A, A_\cap B)$ is smooth and $X=A^\cup B$, then triads $(X;A,B)$ and $(\bar{X};\bar{A},\bar{B})$ have the same homotopy type.*
ii) *If $\bar{A}$ is a retract of $A\times I^1$, then $(A, A_\cap B)$ is smooth, and a combinatorial pair (K, L) is also smooth.*
iii) *Let $\phi:(X,A)\to(Y,B)$ be a mapping such that $\phi|X-A$ is homeomorphism onto $Y-B$, and if (X,A) is smooth then (Y,B) is also smooth.*
iv) *If $(A, A_\cap B)$ is smooth then $(\bar{A},(A_\cap B)\times(1))$ and $(A, A_\cap B)$ have the same homotopy types.*
v) *If (X, A) is smooth, then $(X\times I^1, X\times\dot{I}^1{}^\cup A\times I^1)$ and $(X\times I^1, X\times(0)^\cup A\times I^1)$ are also smooth.*
From the lemma we have (Cf. [3])

Theorem (1.26) *If $(A, A_\cap B)$ is smooth and m-connected, $(B, A_\cap B)$ is n-connected, $A_\cap B$ is r-connected and $X=A^\cup B$, then*
i) *$(X;A,B)$ is $(m+n)$-connected,*
ii) *the induced homomorphisms $\phi^*:\pi_p(A, A_\cap B)\to\pi_p(\tilde{A},x_0)$ are onto for $p\leqq m+n+1$ and isomorphic for $p\leqq m+n$,*
iii) *and the injection homomorphisms $i^*:\pi_p(A, A_\cap B)\to\pi_p(X,B)$ are isomorphisms into for $p\leqq m+r$ and their image are direct factors of $\pi_p(X,B)$, and we have $\pi_p(X,B)\approx\pi_p(A,A_\cap B)\oplus\pi_p(X;A,B)$.*

Let $\chi_i:(E^n,S^{n-1},y_*)\to(X^*,X,x_0)$ be mappings such that $\chi_i|E^n-S^{n-1}$ are homeomorphisms, $\underset{i}{\cup}\chi_i(E^n-S^{n-1})=X^*-X$ and $\chi_i(E^n-S^{n-1})$ are disjoint to each other. The mappings χ_i will be refered to us ***characteristic maps***, and we donote $\mathcal{E}_i^n = \chi_i(E^n)$, $\dot{\mathcal{E}}_i^n = \chi_i(S^{n-1})$, $\mathcal{E}^n = \bigcup_i \mathcal{E}_i^n$ and $\dot{\mathcal{E}}_i^n = \bigcup \dot{\mathcal{E}}_i^n$. By iii) of (1.25), $(\mathcal{E}^n, \dot{\mathcal{E}}^n)$ is smooth and the theorem (1.26) is available for the triad $(X^*; \mathcal{E}^n, X)$.

Set $E_{\frac{1}{2}}^n = \{(x_1, \dots, x_{n+1}) | \sum_i^n = 1/4\}$, $\sigma_i^n = \chi_i(E_{\frac{1}{2}}^n \cup [1/2, 1])$, $Y_i = \mathcal{E}_i^n - \text{Int.}\ \sigma_i^n$, $\sigma^n = \bigcup_i \sigma_i^n$ and $Y = \bigcup_i Y_i$. The pairs $(\mathcal{E}^n, \dot{\mathcal{E}}^n)$ and $(\mathcal{E}^n, Y)$ have the same homotopy type, and in the exact sequence $\cdots \to \pi_p(\sigma^n, \dot{\sigma}^n) \xrightarrow{i^*} \pi_p(\mathcal{E}^n, Y) \to \pi_p(\mathcal{E}^n; \sigma^n, Y) \to \pi_{p-1}(\mathcal{E}^n, \dot{\sigma}^n) \dots$, i^* is equivalent to a homomorphism: $\chi^*: \sum_i \pi_p(E^n, S^{n-1}) \to \pi_p(\mathcal{E}^n, \dot{\mathcal{E}}^n)$ which is given by $\chi^*(\alpha_1 + \cdots + \alpha_i + \cdots) = \chi_1^*(\alpha_1) + \cdots + \chi_i^*(\alpha_i) + \cdots$. From i) and iii) of (1.26) we have

Corollary. (1.27) $\chi^*: \sum_i \pi_p(E^n, S^{n-1}) \to \pi_p(\mathcal{E}^n, \dot{\mathcal{E}}^n)$ *are isomoprphisms into and* $\pi_p(\mathcal{E}^n, \dot{\mathcal{E}}^n) \approx \sum_i \pi_p(E^n, S^{n-1}) \oplus \pi_p(\mathcal{E}^n; \sigma^n, Y)$ *for* $p \leq 2n-3$, *and if* $\dot{\mathcal{E}}^n$ *is m-connected* χ^* *is onto for* $p \leq n + Min.(m, n-1) - 1$.

Chapter 2. Suspension, Products and Hopf costruction.

i) Suspension

Let $d: (X \times E^1, X \times S^0 \cup x_0 \times E^1) \to (E(X), x_0)$ be a map identifying the subset $X \times S^0 \cup x_0 \times E^1$ to the single point x_0, and denote $\hat{X}_+ = d(X \times [0,1])$ and $\hat{X}_- = d(X \times [-1, 0])$. $E(X)$ is called *a suspended space of* X, and we identify the point x of X to the point $(x, 0)$ of $E(X)$, thus $E(X) = \hat{X}_+ \cup \hat{X}_-$ and $X = \hat{X}_+ \cap \hat{X}_-$. S^{n+1} is a suspended space of S^n with respect to the shrinking map d_n of (1.4).

Define *a sum* $f+g$ *of* f *and* $g: (E(X), x_0) \to (Y, y_0)$ and *an invers* $-f$ of f by

$$
\begin{aligned}
(2.1) \qquad (f+g)(d(x,t)) &= f(d(x, 2t-1)) && 0 \leq t \leq 1, \\
&= g(d(x, 2t+1)) && -1 \leq t \leq 0, \\
(-f)(d(x,t)) &= f(d(x, -t)) && -1 \leq t \leq 0,
\end{aligned}
$$

then the homotopy classes of f form a group, which coincide to the fundamental group of the function space $Y_0^X = \{f: X \to Y | f(x_0) = y_0\}$, with reference point $f_0: X \to y_0$.

A suspension (map) $Ef: (I^{n+1}, \dot{I}^{n+1}) \to (E(X), x_0)$ *of* $f: (I^n, \dot{I}^n) \to (X, x_0)$ is defined by

$$
(2.2) \qquad Ef(x_1, \dots, x_{n+1}) = d(f(x_1, \dots, x_n), 2x_{n+1} - 1),
$$

then we have $E(f +_i g) = Ef +_i Eg$ and $E(-_i f) = -_i Ef$ for $1 \leq i \leq n$, and therefore we obtain the suspension homomorphism

$$
(2.3) \qquad E: \pi_n(X, x_0) \to \pi_{n+1}(E(X), x_0).
$$

For $f: (I^n, \dot{I}^n) \to (X, x_0)$ and $g_1, g_2: (E(X), x_0) \to (Y, y_0)$ we have

$$
(2.4) \quad (g_1 + g_2) \circ Ef = g_1 \circ Ef +_{n+1} g_2 \circ Ef \ \textit{and} \ (-g) \circ Ef = -_{n+1}(g \circ Ef).
$$

Since $\hat{X}_+$ and $\hat{X}_-$ are cortractible, the exactness of the homotopy sequences of the pairs $(\hat{X}_+, X, x_0)$ and $(E(X), \hat{X}_-, x_0)$ lead that the homomorphisms $\partial : \pi_{n+1}(\hat{X}_+, X, x_0) \to \pi_n(X, x_0)$ and $j^* : \pi_n(E(X), x_0) \to \pi_n(E(X), \hat{X}_-, x_0)$ are isomorphisms onto. Consider the diagram

(2.5)
$$\begin{tikzcd}
\cdots \pi_{n+1}(E(X); \hat{X}_+, \hat{X}_-) \arrow[r, "\beta_+"] \arrow[dr, "\Delta"'] & \pi_n(\hat{X}_+, X) \arrow[r, "i^*"] \arrow[d, "\partial"] & \pi_n(E(X), \hat{X}_-) \arrow[r, "j^*"] & \pi_n(E(X); \hat{X}_+, \hat{X}_-) \to \cdots \arrow[dl, dashed, "I"] \\
& \pi_{n-1}(X) \arrow[r, "E"] & \pi_n(E(X)) \arrow[u, "j^{*\prime}"'] &
\end{tikzcd}$$

where $\Delta = \partial \circ \beta_+$ and $I = j^{*\prime} \circ j^*$. It is easily verified that $E = j^*_{-1} \circ i^* \circ \partial^{-1}$ and that the sequence of the homomorphisms $\cdots \xrightarrow{\Delta} \xrightarrow{E} \xrightarrow{I} \cdots$ is exact. By (1.23), v) of (1.25) and i) of (1.26),

(2.6) *if X is r-connected and smooth, then $(E(X|, \hat{X}_+, \hat{X}_-)$ is $(2r+2)$-connected and therefore the suspension homomorphisms $E : \pi_n(X) \to \pi_{n+1}(E))$ are isomorphic for $n \leqq 2r$ and onto for $n = 2r+1$.*

Note that

(2.7) *if $f(I^{n+1}, \dot{I}^{n+1}) \to (I^{r+1}, \dot{I}^{r+1})$ is a map snch that $\varepsilon_n \circ (f \,|\, \dot{I}^{n+1})$ is a representative of an element $\alpha \in \pi_n(S^r)$ and if $g : (I^{r+1}, \dot{I}^{r+1}) \to (X_0 x_0)$ is a representative of $\beta \in \pi_{r+1}(X)$, then the composite mapping $g \circ f : (I^{n+1}, \dot{I}^{n+1}) \to (X, x_0)$ represents $\beta \circ E(\alpha) \in \pi_n(X, x_0)$.*

Define a mapping $D^n : (X \times I^n, X \times \dot{I}^n \cup x_0 \times I^n) \to (E^n(X), x)$ inductively by setting $D^1 = d$ and $D^n(x, (x_1, \dots, x_n)) = d(D^{n-1}(x, (x_1, \dots, x_{n-1}), 2x_n - 1)$, where $E^n(X)$ indicates the n-fold suspended space of X. Since D^n maps $X \times I^n - (X \times \dot{I}^n \cup x_0 \times I^n)$ homeomorphically onto $E^n(X) - x_0$, we can define a mapping $\phi_n : (X \times S^n, X \vee S^n) \to (E^n(X), x_0)$ such that

$$\phi_n(x, \psi_n(y)) = D^n(x, y) \qquad x \in X, \ y \in I^n. \tag{2.8}$$

Define a product $f \times g : (I^{p+q}, \dot{I}^{p+q}) \to (A \times B, A \vee B)$ of $f : (I^p, \dot{I}^p) \to (A, a_0)$ and $g : (I^q, \dot{I}^q) \to (B, b_0)$ by

$$(f \times g)(x_1, \dots, x_{p+q}) = (f(x_1, \dots, x_p), g(x_{p+1}, \dots, x_{p+q})), \tag{2.9}$$

then $f \simeq f'$, $g \simeq g'$ implies $f \times g \simeq f' \times g'$ and hence a product $\alpha \times \beta \in \pi_{p+q}(A \times B, A \vee B)$ of $\alpha \in \pi_p(A)$ and $\beta \in \pi_q(B)$ is defined. If $f : (I^m, \dot{I}^m) \to (X, x_0)$ is a representatve of $\alpha \in \pi_n(X)$, we have by (2.8) $\phi_n(f \times \psi_n)(x, (y_1, \dots, y_n)) = D^n(f(x), (y_1, \dots, y_n) = d(D^{n-1}(f(x), (y_1, \dots, y_{n-1}), 2y_n - 1) = ED^{n-1}(f(x), (y_1, \dots, y_{n-1})) = \cdots = E^n f(x)$, in which E^n indicates the n-fold suspension. Therefore

(2.10) *$\phi_n^*(\alpha \times \iota_n) = E^n(\alpha)$, when ι_n[9] is the generator of $\pi_n(S^n)$ represented by ψ_n.*

Finally we remark that the suspension Ef of $f : (I^n, \dot{I}^n) \to (X, x_0)$ satisfies the condition $f(I_+^{n+1}) \subset \hat{X}_+$, $f(I_-^{n+1}) \subset \hat{X}_-$ and $Ef\left(x_1, \dots, x_n, \frac{1}{2}\right) = f(x_1, \dots, x_n)$,

9) This notation: $\iota_n \in \pi_n(S^n)$ will be used through the paper.

and that if a map $F:(I^{n+1}, \dot{I}^{n+1}) \to (E(X), x_0)$ satisfies this condition then we have $Ef \simeq F$.

ii) Products.

The original *product* of J. H. C. Whitehead $[f, g]:(\dot{I}^{p+q}, 0) \to (X, x_0)$ of $f:(I^p, \dot{I}^p) \to (X, x_0)$ and $g:(I^q, \dot{I}^q) \to (X, x_0)$ is given by

$$(2.11) \qquad \begin{aligned} [f, g](x_1, \dots, x_{p+q}) &= f(x_1, \dots, x_p) && (x_{p+1}, \dots, x_{p+q}) \in \dot{I}^q, \\ &= g(x_{p+1}, \dots, x_{p+q}) && (x_1, \dots, x_p) \in \dot{I}^p, \end{aligned}$$

or

$$\begin{aligned} [f, g](\phi_{p,q}(x, y, t)) &= f((1-t)x_1, \dots, (1-t)x_p) && 0 \leq t \leq 1, \\ &= g((1+t)y_1, \dots, (1+t)y_q) && -1 \leq t \leq 0. \end{aligned}$$

If f_t and g_t are homotopies, then $[f_t, g_t]$ is a homotopy from $[f_0, g_0]$ to $[f_1, g_1]$ and therefore the product $[\alpha, \beta] \in \pi_{p+q-1}(X, x_0)$ of $\alpha \in \pi_p(X, x_0)$ and $\beta \in \pi_q(X, x_0)$ can be defined. Let $i_1: A \to A \vee B$ and $i_2: B \to A \vee B$ are injections such that $i_1(a)=(a, b_0)$ and $i_2(b)=(a_0, b)$. By (2.9) and (2.11) we have for $f:(I^p, \dot{I}^p) \to (A, a_0)$ and $g:(I^q, \dot{I}^q) \to (B, b_0)$

$$(2.12) \qquad [i_1 \circ f, i_2 \circ g] = f \times g | \dot{I}^{p+q}.$$

Let $f:(S^p, y) \to (X, x_0)$ and $g:(S^q, y_*) \to (X, x_0)$ be representatives of $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$ respectively and let $f \vee g:(S^p \vee S^q, y_* \times y_*) \to (X, x_0)$ be a mapping such that $(f \vee g)(x, y_*) = f(x)$ and $(f \vee g)(y_*, x') = g(x')$ for $x \in S^p$ and $x' \in S^q$. Then the composite map

$$(2.13) \qquad (f \vee g) \circ \psi_{p,q}:(\dot{I}^{p+q}, 0_*) \to (S^p \vee S^q, y_* \times y_*) \to (X, x_0)$$

is a representative of $[\alpha, \beta]$.

Now we define *a (relative) product* $[\alpha, \beta]_r \in \pi_{p+q-1}(X, A, x_0)$ ot $\alpha \in \pi_p(A, x_0)$ and $\beta \in \pi_q(X, A, x)$. Let $f:(I^p, \dot{I}^p) \to (A, x_0)$ and $g(I^q, I^{q-1}, J^{q-1}) \to (X, A, x_0)$ be representatives of α and β respectively. Define a mapping $(f, g)_r:(J^{p+q-1}, \dot{I}^{p+q-1}, 0_*) \to (x, A, x_0)$ by

$$(2.14)' \qquad \begin{aligned} (f, g)_r(x_1, \dots, x_{p+q}) &= f(x_1, \dots, x_p) && (x_{p+1}, \dots, x_{p+q}) \in J^{q-1}, \\ &= g(x_{p+1}, \dots, x_{p+q}) && (x_1, \dots, x_p) \in \dot{I}^p, \end{aligned}$$

and define a relative product $[f, g]_r:(I^{p+q-}, \dot{I}^{p+q-1}, 0_*) \to (X, A, x_0)$ of f and g by

$$(2.14) \qquad [f, g]_r = (f, g)_r \circ P_{p+q-1}.$$

$f \simeq f'$ and $g \simeq g'$ imply $[f, g]_r \simeq [f', g']_r$ and $[f, g]_r$ is a representative of the relative product $[\alpha, \beta]_r$.

Next we define *a (triad) product* $[\alpha, \beta]_t \in \pi_{p+q-1}(X; A, B, x_0)$ of $\alpha \in \pi_p(B, A \cap B, x_0)$ and $\beta \in \pi_q(A, A \cap B, x_0)$. Let $f:(I^p, I^{p-1}, J^{p-1}) \to (B, A \cap B, x_0)$ and $g:(I^q, I^{q-1}, J^{q-1}) \to (B, A \cap B, x_0)$ be representatives or α and β respectively. Define a mapping $(f, g)_t:(K^{p+q-1}; J^{p+q-2}, J_0^{r+q-2}, 0_*) \to (X; A, B, x_0)$ by

(2.15)′ $\langle f, g\rangle_t(x_1, \dots, x_{p+q}) = f(x_1, \dots, x_{p-1}, x_{p+q}) \qquad (x_p, \dots, x_{p+q-1}) \in J^{q-1},$
$\qquad = g(x_p, \dots, x_{p+q-1}) \qquad (x_1, \dots, x_{p-1}, x_{p+q}) \in J^{p-1},$

and define *a triad product* $[f, g]_t : (I^{p+q-1}; J^{p+q-2}, J_0^{p+q-2}, 0) \to (X; A, B, x_0)$ of f and g by

$$(2.15)\qquad [f, g]_t = \langle f, g\rangle_t \circ P'^{-1}_{p+q-1},$$

$f \simeq f'$ and $g \simeq g'$ imply $[f, g]_t \simeq [f', g']_t$ and $[f, g]_t$ is a representative of $[\alpha, \beta]_t$.

We have

(2.16) $[\alpha, \beta_1+\beta_2]=[\alpha, \beta_1]+[\alpha, \beta_2] \quad \alpha \in \pi_p(X), \beta_1, \beta_2 \in \pi_q(X) \quad (q>1),$
$[\alpha_1+\alpha_1, \beta]=[\alpha_1, \beta]+[\alpha_2, \beta] \quad \alpha_1, \alpha_2 \in \pi_p(X), \beta \in \pi_q(X) \quad (p>1),$
$[\alpha, \beta_1+\beta_2]_r=[\alpha, \beta_1]_r+[\alpha, \beta_2]_r \quad \alpha \in \pi_p(A), \beta_1, \beta_2 \in \pi_q(X, A) \quad (q>2),$
$[\alpha_1+\alpha_2, \beta]_r=[\alpha_1, \beta]_r+[\alpha_2, \beta]_r \quad \alpha_1, \alpha_2 \in \pi_p(A), \beta \in \pi_q(X, A) \quad (p>1),$
$[\alpha, \beta_1+\beta_2]_t=[\alpha, \beta_1]_t+[\alpha, \beta_2]_t \quad \alpha \in \pi_p(B, A \cap B), \beta_1, \beta_2 \in \pi_q(A, A \cap B) \quad (q>2),$
$[\alpha_1+\alpha_2, \beta]_t=[\alpha_1, \beta]_t+[\alpha_2, \beta]_t \quad \alpha_1, \alpha_2 \in \pi_p(B, A \cap B), \beta \in \pi_q(A, A \cap B) \quad (p>2),$

(2.17) $f^*[\alpha, \beta] = [f^*(\alpha), f^*(\beta)] \quad \alpha \in \pi_p(X), \beta \in \pi_q(X),$
$f^*[\alpha, \beta]_r = [f_1^*(\alpha), f^*(\beta)]_r \quad \alpha \in \pi_p(A), \beta \in \pi_q(X, A),$
$f^*[\alpha, \beta]_t = [f_2^*(\alpha), f_1^*(\beta)]_t \quad \alpha \in \pi_p(B, A \cap B), \beta \in \pi_q(A, A \cap B),$

for $f : (X; A, B, x_0) \to (Y; C, D, y_0)$, $f_1 = f \mid A$ and $f_2 = f \mid B$.

From the definitions of products and boundaries, we have $\partial[f, g]_r = [f, \partial g]$ and $\beta_+[f, g]_t = [\partial f, g]_r$, and we have

$$(2.18)\qquad \partial[\alpha, \beta]_r = [\alpha, \partial\beta] \quad \textit{and} \quad \beta_+[\alpha, \beta]_t = [\partial\alpha, \beta]_r.$$

Next consider a product $[\alpha, j^*(\beta)]$, where $\alpha \in \pi_p(A)$, $\beta \in \pi_q(X)$ and j^* is the natural homomorphism $: \pi_q(X) \to \pi_q(X, A)$. Let $f : (I^p, \dot{I}^p) \to (A, x_0)$ and $g : (I^q, \dot{I}^q) \to (X, x_0)$ be representatives of α and β respectively, then the map $[f, g]_r : (I^{p+q-1}, \dot{I}^{p+q-1}, 0) \to (X, A, x_0)$ represents $[\alpha, j^*(\beta)]_r$. Remark that if a mapping $F : (\dot{I}^{p+q}, 0_*) \to (X, x_0)$ represents $\gamma \in \pi_{p+q-1}(X)$ and $F(J^{p+q-1}) \subset A$, then $F \mid I^{p+q-1} : (I^{p+q-1}, \dot{I}^{p+q-1}, 0_*) \to (X, A, x_0)$ represents $j^*(\gamma) \in \pi_{p+q-1}(X, A)$. Making use of this remark, we have $[\alpha, j^*(\beta)]_r = j^*(\gamma)$ where γ is represented by a mapping $F : (\dot{I}^{p+q}, 0_*) \to (X, x_0)$ such that $F \mid I^{p+q-1} = [f, g]_r$ and $F \mid J^{p+q-1} = ([f, g] \mid I^{p+q-1}) \circ P_n$. Since $[f, g]_r = ([f, g] \mid J^{p+q-1}) \circ P_n^{-1}$, we have $F = [f, g] \circ \bar{P}_n$ where $\bar{P}_n$ is given by $\bar{P}_n \mid I_n = P_n$ and $\bar{P}_n \mid J_n = P_n^{-1}$ and hence $\bar{P}_n$ is a homeomorphism reversing the orientation. Consequently we obtain

$$(2.19)\qquad j^*[\alpha, \beta] = -[\alpha, j^*(\beta)]_r \quad \textit{for } \alpha \in \pi_p(A) \text{ and } \beta \in \pi_q(X).$$

Similarly we have

$$(2.19)'\qquad j_0^*[\alpha, i^*(\beta)]_r = (-1)^q[j^*(\alpha), \beta]_t \quad \textit{for } \alpha \in \pi_p(B),\ \beta \in \pi_q(A, A \cap B)$$

and for the natural homomorphisms $j^* : \pi_p(B) \to \pi_p(B, A \cap B)$, $i^* : \pi_q(A, A \cap B) \to$

$\pi_q(X, B)$ and $j_0^*: \pi_{p+q-1}(X;B)\to\pi_{p+q-1}(X;A,B)$.

Let $\eta: I^p\times I^q\to I^{p+q}$ be a mapping given by $\eta(x_1,\dots,x_p, y_1,\dots,y_q)=(x_1,\dots,x_{p-1}, y_1,\dots,y_q,x_p)$, then the mapping $(f,g)_t;(K^{p+q-1};J^{p+q-2},J_0^{p+q-2},0_*)\to(X;A,B,x_0)$ of (2.15)' satisfies the condition:

(2.20) $(f,g)_t(\eta(I^p\times J^{q-1}))\subset B$, $(f,g)_t(\eta(J^{p-1}\times I^q))\subset A$, $(f,g)_t(\eta(\dot I^p\times J^{q-1}\cup J^{p-1}\times\dot I^q))\subset A\cap B$, and $(f,g)_t|\eta(I^p\times 0_*)$ and $(f,g)_t|\eta(0^*\times I^q)$ represent the elements $\alpha\in\pi_p(B,A\cap B)$ and $\beta\in\pi_q(A,A\cap B)$ respectively.

Lemma (2.21) *If a mapping* $F:(K^{p+q-1},J^{p+-1},J_0^{p+q-1},0_*)\to(X;A,B,x_0)$ *satisfies the condition then the composite map* $F\circ P'_{p+q+1}$ *represents* $[\alpha,\beta]_t$.

The proof of the lemma follows the fact that $\dot I^p\times 0_*\cup 0_*\times\dot I^q$, $\dot I^p\times J^{q-1}\cup I^p\times 0_*$ and $0_*\times I^q\cup J^{p-1}\times\dot I^q$ are retacts of $\dot I^p\times J^{q-1}\cup J^{p-1}\times\dot I^q$, $I^p\times J^{q-1}$ and $J^{p-1}\times I^q$ respectively.

iii) Join and Hopf construction.

A join $f*g:(\dot I^{p++2},0_*)\to(\dot I^{m+n+2},0)$ of $f:(\dot I^{p+1},0_*)\to(\dot I^{m+1},0_*)$ and $g:(\dot I^{+1},0_*)\to(\dot I^{n+1},0_*)$ is defined by

(2.21) $$(f*g)(\Phi_{p,q}(x,y,t))=\Phi_{m,n}(f(x),g(y),t).$$

Let $\bar f:(I^{p+q},\dot I^{p+1})\to(I^{m+1},\dot I^{m+1})$ and $\bar g:(I^{p+1},\dot I^{p+1})\to(I^{n+1},\dot I^{n+1})$ be extensions of $f=\bar f|\dot I^{p+1}$ and $g=\bar g|\dot I^{q+1}$ such that if $f(x_1,\dots,x_{p+1})=(x'_1,\dots,x'_{m+1})$ and $g(y_1,\dots,y_{q+1})=(y'_1,\dots,y'_{n+1})$ then $\bar f(tx_1,\dots,tx_{p+1})=(tx'_1,\dots,tx'_{m+1})$ and $\bar g(ty'_1,\dots,ty_{q+1})=(ty'_1,\dots,ty'_{n+1})$. Define a mapping $\bar f\times\bar g:(I^{p+q+2},\dot I^{p+q+2})\to(I^{m+n+2},\dot I^{m+n+2})$ by $(\bar f\times\bar g)(x_1,\dots,x_{p+q+2})=(f(x_1,\dots,x_{p+1}), g(x_{p+2},\dots,x_{p+q+2}))$, then we have $\partial\bar f=f$, $\partial\bar g=g$ *and* $\partial(\bar f\times\bar g)=f*g$.

As is shown in [22], the join operator is induced in homotopy groups and is bilinear. Let $\alpha\in\pi_p(S^m)$ and $\beta\in\pi_q(S^n)$ be the classes of $\varepsilon_m\circ f$ and $\varepsilon_n\circ g$ respectively, then $\psi_{m+1}\circ\bar f:(I^{p+1},\dot I^{p+1})\to(S^{m+1},y_*)$ and $\psi_{n+1}\circ\bar g:(I^{q+1},\dot I^{q+1})\to(S^{n+1},y_*)$ represent $E(\alpha)$ and $E(\beta)$ by (2.7). From (2.9) and (1.11) $\phi_{m+1,n+1}((\psi_{n+1}\circ\bar f)\times(\psi_{m+1}\circ\bar g))=\phi_{m+1,n+1}\circ\psi_{m+1,n+1}(\bar f\times\bar g)=\psi_{m+n+2}(\bar f\times\bar g):(I^{p+q+2},\dot I^{p+q+2})\to(S^{m+n+1},y_*)$, and by (2.7) we have

(2.22) $$\phi^*_{m+1,n+1}(E(\alpha)\times E(\beta))=E(\alpha*\beta)\qquad \alpha\in\pi_p(S^m),\beta\in\pi_q(S^n).$$

Let $f,g,\bar f$ and $\bar g$ be mappings as above. For two mappings $f':(I^{m+1},\dot I^{m+1})\to(X,x_0)$ and $g':(I^{n+1},\dot I^{n+1})\to(X,x_0)$,

$$\begin{aligned}[f',g']\circ(f*g)(\Phi_{p,q}(x,y,t)) &= [f',g'](\Phi_{m,n}(f(x),g(y),t))\\ &= f'(\bar f((1-t)x_1,\dots,(1-t)x_{p+1}) \qquad 0\leqq t\leqq 1,\\ &= g'(\bar g((1+t)y_1,\dots,(1+t)y_{q+1}) \qquad -1\leqq t\leqq 0.\\ &= [f'\circ\bar f,g'\circ\bar g](\Phi_{p,q}(x,y,t)).\end{aligned}$$ Therefore by (2.7) we have

(2.23) $$[\alpha',\beta']\circ(\alpha*\beta)=[\alpha'\circ E(\alpha),\beta'\circ E(\beta)],$$

where $\alpha'\in\pi_{m+1}(X)$, $\beta'\in\pi_{n+1}(X)$, $\alpha\in\pi_p(S^m)$ *and* $\beta\in\pi_q(S^n)$.

The following property of join was provided in [22]

(2.24) $(-)^{(n+1)(r+1)}\iota_n * \alpha = \alpha * \iota_n = E^{n+1}(\alpha)$, *where* $\alpha\in\pi_m(S^r)$ *and* $\iota_n\in\pi_n(S^n)$ *is represented by the identity map.*

A Hopf construction $Gf:(\dot{I}^{p+q+2},0_*)\to(E(X),x_0)$ of $f:(\dot{I}^{p+1}\times\dot{I}^{q+1},0_*)\to(X,x_0)$ is defined by

$$(2.25)\qquad Gf(\Phi_{p,q}(x,y,t)) = d(f(x,y),t).$$

The mapping Gf satisfies the conditions $Gf(I^{p+1}\times\dot{I}^{q+1})\subset\hat{X}_+$, $Gf(\dot{I}^{p+1}\times I^{q+1})\subset\hat{X}_-$ and $Gf|\dot{I}^{p+1}\times\dot{I}^{q+1}=f$, and conversely, any mapping $G^1:(\dot{I}^{p+q+2},0_*)\to(E(X),x_0)$ satisfying the condition is homotopic to Gf.

We say that the map f has a type (α,β) if $f|\dot{I}^{p+1}\times 0_*$ and $f|0_*\times\dot{I}^{q+1}$ represent $\alpha\in\pi_p(X)$ and $\beta\in\pi_q(X)$ respectively. Also a mapping $f':(S^p\times S^q,y_*\times y_*)\to(X,x_0)$ is said to have a type (α,β), if $f'|S^p\times y_*$ and $f'|y_*\times S^q$ represent $\alpha\in\pi_q(X)$ and $\beta\in\pi_q(X)$ respectively. Consider the composite map $f'\circ\psi_{p,q}$: $(\dot{I}^{p+q},0_*)\to(X,x_0)$, then $f'\circ\psi_{p,q}|\dot{I}^{p+q}$ represents $[\alpha,\beta]$ by (2.13), hence $f'\circ\psi_{p,q}$ gives a nullhomotopy of $f'\circ\psi_{p,q}|\dot{I}^{p+q}$ and $[\alpha,\beta]=0$. Conversely if $[\alpha,\beta]=0$, there is am apping $F:(I^{p+q},0_*)\to(X,x_0)$ such that $F|\dot{I}^{p+q}=f'\circ\psi_{p,q}|\dot{I}^{p+q}$, $f':(S^p\vee S^q,y_*\times y_*)\to(X,x_0)$ and $f'|S^p\times y_*$ and $f'|y_*\times S^q$ represent α and β respectively. Define $f'|(S^p\times S^q-S^p\vee S^q)$ by setting $f'(x)=F\circ\psi_{p,q}^{-1}(x)$ for $x\in S^p\times S^q-S^p\vee S^q$, then f' has the type $[\alpha,\beta]$.

(2.26) *There is a mapping* $f:\dot{I}^{p+1}\times\dot{I}^{q+1}\to X$ *of type* (α,β) *if and only if* $[\alpha,\beta]=0$.

Since $\hat{X}_+$ and $\hat{X}_-$ are contractible the boundary homomorphisms ∂_+: $\pi_{q+1}(\hat{X}_+,X)\to\pi_q(X)$ and $\partial_-:\pi_{p+1}(X_-,X)\to\pi_p(X)$ are isomorphisms onto. Let $\bar{\alpha}\in\pi_{p+1}(\hat{X}_-,X)$ and $\bar{\beta}\in\pi_{q+1}(\hat{X}_+,X)$ be elements such that $\partial\bar{\alpha}=\alpha\in\pi_p(X)$ and $\partial\bar{\beta}=\beta\in\pi_q(X)$, then $\Delta[\bar{\alpha},\bar{\beta}]_t=\partial[\alpha,\bar{\beta}]_r=[\alpha,\beta]$. The exactness of the sequence

$$\cdots\to\pi_{p+q+1}(E(X))\xrightarrow{I}\pi_{p+q+1}(E(X);\hat{X}_+,\hat{X}_-)\xrightarrow{\Delta}\pi_{p+q-1}(X)\xrightarrow{E}\pi_{p+q}(E(X))\to\cdots$$

leads

$$(2.27)\qquad E[\alpha,\beta]=0\quad \alpha\in\pi_p(X),\ \beta\in\pi_q(X).$$

If $[\alpha,\beta]=0$, then $\Delta[\bar{\alpha},\bar{\beta}]_t=0$ and there is an element γ of $\pi_{p+q+1}(E(X))$ such that $I(\gamma)=[\bar{\alpha},\bar{\beta}]_t$.

Lemma (2.28) $I(\gamma)=[\bar{\alpha},\bar{\beta}]_t$ *if and only if* $(-1)^{p(q+1)+1}\gamma$ *is represented by the Hopf construction of a mapping*: $(\dot{I}^{p+1}\times\dot{I}^{q+1},0_*)\to(X,x_0)$ *of the type* (α,β).

First remark that if a mapping $F:(\dot{I}^{n+1},0_*)\to(X,x_0)$ represents $\gamma\in\pi_n(X)$, and if $F(I^n_n)\subset B$ and $F(I^n)\subset A$, then $(F|K^n)\circ P'^{-1}_n:(I^n;J^{n-1},I^{n-1},0_*)\to(X;A,B,x_0)$ represents $-I(\gamma)$, where $I:\pi_n(X)\to\pi_n(X;A,B)$ is the natural homomorphism.

Let $\bar{f}:(I^{p+1},I^p,J^p)\to(\hat{X}_-,X,x_0)$ and $\bar{g}:(I^{q+1},I^q,J^q)\to(\hat{X}_+,X,x_0)$ be representatives of $\bar{\alpha}$ and $\bar{\beta}$ respectively. We extend the mapping $(\bar{f}, \bar{g})_t : (K^{p+q+1}; J^{p+q}, J^{p+q}, 0_*) \to (E(X); X_+, X_-, x_0)$ of $(2.14)'$ over $\dot{I}^{p+q+2}$ as follows, and obtain a map $F: \dot{I}^{p+q+2} \to E(X)$. Since $((\bar{f}, \bar{g})_t | \dot{I}^{p+q}) \circ P' = \partial\beta_+[\bar{f}, \bar{g}]_t = [f, g] \simeq 0$, the mapping $F | \dot{I}^{p+q} = (f, g)_t | \dot{I}^{p+q}$ is extendable over I^{p+q} such that $F(I^{p+q}) \subset X$. Since $\hat{X}_+$ and $\hat{X}_-$ are contractible, the mappings $F | \dot{I}^{p+q+1} : \dot{I}^{p+q+1} \to \hat{X}_+$ and $F | \dot{I}^{p+q+1}_{p+q+1} : \dot{I}^{p+q+1}_{p+q+1} \to \hat{X}_-$ are extendable over I^{p+q+1} and I^{p+q+1}_{p+q+1} such that $F(I^{p+q+1}) \subset \hat{X}_+$ and $F(I^{p+q+1}_{p+q+1}) \subset X_-$. Let $\eta : I^{p+q+2} \to \dot{I}^{p+q+2}$ be a mapping of degree $(-1)^{p(q+1)}$ given by $\eta(x_1, \dots, x_{p+q+2}) = (x_{p+1}, \dots, x_{p+q+1}, x_1, \dots, x_p, x_{p+q+2})$. Then the composite map $F \circ \eta : \dot{I}^{p+q+2} \to E(X)$ maps subsets $I^{p+1} \times \dot{I}^{q+1}$ and $\dot{I}^{p+1} \times \dot{I}^{q+1}$ into $\hat{X}_+$ and $\hat{X}_-$ respectively, and therefore $F \circ \eta$ is homotopic to the Hopf construction of the mapping $F \circ \eta | \dot{I}^{p+1} \times \dot{I}^{q+1}$ which has type (α, β). By making use of the above remark, the necessity of the lemma is established.

Conversely, let $F' : \dot{I}^{p+q+2} \to X$ be the Hopf construction of $F' | \dot{I}^{p+1} \times \dot{I}^{q+1}$, then $F' \circ \eta$ maps I^{p+q+1} and I^{p+q+1}_{p+q+1} into $\hat{X}_+$ and $\hat{X}_-$ respectively, and therefore $(F' \circ \eta | K^{p+q+1}) \circ P'_{p+q+1}$ represents $(-1) I(\{F'\})$. While $F' \circ \eta | K^{p+q+1}$ satisfies the condition (2.20) and homotopic to $(\bar{f}, \bar{g})_t$, and the sufficiency of the lemma is established.

Define *a suspension* $E'f : (I^{n+1}, I^n, J^n) \to (E(X), E(A), x_0)$ of $f : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$ by

$$(2.29) \qquad E'f(x_1, \dots, x_{n+1}) = d(f(x_1, \dots, x_{n-1}, x_{n+1}), 2x_n - 1).$$

Clearly $f \simeq f'$ implies $E'f \simeq E'f'$, and $E'(f +_i g) = E'f +_i E'g$ $(1 \leqq i \leqq n-1)$, and we obtain a suspension homomorphism $E' : \pi_n(X, A) \to \pi_{n+1}(E(X), E(A))$. Also we have $\partial(E'f) = E(\partial f)$ and $E(\alpha) = -E'(j^*(\alpha))$ for $\alpha \in \pi_n(X)$.

Now we shall prove the fact analogeous to (2.27):

$$(2.30) \qquad E'[\alpha, \beta]_r = 0 \quad \textit{for } \alpha \in \pi_p(A) \textit{ and } \beta \in \pi_q(X, A).$$

Set $J^n_+ = \{x \in J^n | x_n \geqq 1/2\}$, $J^n_- = \{x \in J^n | x_n \leqq 1/2\}$, $\dot{I}^n_+ = \dot{I}^n \cap J^n_+$, and $\dot{I}^n_- = \dot{I}^n \cap J^n_-$. First remark that if a mapping $F : J^{n+1} \to E(X)$ satisfies condition

(2.31) $F(J^{n+1}_+) \subset \hat{X}_+$, $F(J^{n+1}_-) \subset \hat{X}_-$, $F(\dot{I}^n_+) \subset \hat{A}_+$ $F(\dot{I}^n_-) \subset \hat{A}_-$, and if $F_0 : J^n \to X$ is a mapping given by $F_0(x_1, \dots, x_{n+1}) = F(x_1, \dots, x_n, 1/2, x_{n+1})$, and $F_0 \circ P_n$ represents $\alpha \in \pi_n(X, A)$, then $F \circ P_{n+1}$ represents $E'(\alpha)$.

Define subsets of $\dot{I}^{n+1}$ by $L^n = Cl(\dot{I}^{n+1} - I^n_{n-1} - I^{n-n}_n)$, $K^{n-1}_1 = Cl(\dot{I}^n_n - I^{n-1}_{n-1} - I^n)$, $K^{n-1}_2 = Cl(\dot{I}^n_{n-1} - I^n_n - I^n)$, $J^{n-2}_1 = K^{n-1} \cap K^{n-1}_1$ and $J^{n-2}_2 = K^{n-1} \cap K^{n-1}$, then L^n is a closed cell with faces K^{n-1}_1, K^{n-1}_2 and K^{n-1}. There is a homeomorphism $\chi : (J^{p+q}; J^{p+q}_+, J^{p+q}_-, \dot{I}^{p+q}_+, \dot{I}^{p+q}_-) \to (K^{p+q}_1 \cup K^{p+q}_2; K^{p+q}_1, K^{p+q}_2, J^{p+q-1}_1, J^{p+q-1}_2)$ and a mapping $\bar{\chi} : J^{p+q} \times I^1 \to L^{p+q+1}$ such that $\chi(x_1, \dots, x_{p+q-1}, 1/2, x_{p+q-1}) = (x_1, \dots, x_{p+q}, 0)$, $\bar{\chi}(x, 0) = \chi(x)$ for $x \in J^{p+q}$, $\bar{\chi}(x, t) = \chi(x)$ for $x \in \dot{I}^{p+q}$, and $\bar{\chi} | \mathrm{Int.}\, J^{p+q} \times I^1$ is a homeomorphism.

Let $f : (I^p, \dot{I}^p) \to (A, x_0)$ and $g : (I^q, I^{q-1}, J^{q-1}) \to (X, A, x_0)$ be representatives of $\alpha\in\pi_p(A)$ and $\beta\in\pi_q(X, A)$ respectively, and define mappings $\bar{f}:(I^{p+1}, I^p, J^p)\to(\hat{A}_+, A, x_0)$ and $\bar{g}:(I^{q+1}; I^q, I^q_0, K^q)\to(\hat{X}_-; X, \hat{A}_-, x_0)$ by setting $\bar{f}(x_1, \dots, x_{p+1})=d(f(x_1,\dots,x_p), 2x_{p+1}-1)$ and $\bar{g}(x_1,\dots,x_{q+1})=d(g(x_1,\dots,x_{q-1}, x_{q+1}), 2x_q-1)$. Define a mapping $F: L^{p+q+1}\to E(X)$ by

$$\begin{aligned}\bar{F}(x_1,\dots,x_{p+q+2}) &= \bar{f}(x_1,\dots,x_p, x_{p+q+1}) \quad \text{if} \quad (x_{p+1},\dots,x_{p+q}, x_{p+q+2})\in K^q,\\ &= \bar{g}(x_{p+1},\dots,x_{p+q}, x_{p+q+2}) \quad \text{if} \quad (x_1,\dots,x_{p+1}, x_{p+q+1})\in J^p.\end{aligned}$$

The map $F=(\bar{F}\,|\,K_1^{p+q}\cup K_2^{p+q})\circ\chi$ satisfies the condition (2.31) and therefore $F\circ P_{p+q}$ represents $E'[\alpha, \beta]_r$.

By setting $F'_t(x)=\bar{F}(\chi(x, t))$ and $F_t=F'_t\circ P_{p+q}$ we see that $F=F_0$ is homotopic to F_1 which maps J^{p+q} into $E(A)$ and is homotopic to the trivial map, and the proof of (2.30) is established.

Supposed that elements $\alpha\in\pi_p(A)$ and $\beta\in\pi_q(X, A)$ satisfies the condition $[\alpha, \partial\beta]=0$. Consider the following diagram

$$\begin{array}{ccccc} & & \pi_{p+q-1}(X) & \xrightarrow{j'} & \pi_{p+q-1}(X, A) & \xrightarrow{\partial} & \pi_{p+q-2}(A)\\ & & \downarrow E & & \downarrow E' & & \\ \pi_{p+q}(E(A)) & \xrightarrow{i^*} & \pi_{p+q}(E(X)) & \xrightarrow{j} & \pi_{p+q}(E(X), E(A)) & & \end{array}$$

The condition $\partial[\alpha, \beta]_r=[\alpha, \partial\beta]=0$ implies that there is an element γ of $\pi_{p+q-1}(X)$ such that $j'(\gamma)=[\alpha, \beta]_r$. Since $j(E(\gamma))=-E'(j'(\gamma))=-E'[\alpha, \beta]_r=0$ by (2.30), there is an element δ of $\pi_{p+q}(E(A))$ such that $i^*(\delta)=E(\gamma)$.

Lemma (2.32) *With the above hypothesis,* $(-1)^{p(q+1)}\delta$ *is represented by the Hopf construction of a mapping of type* $(\alpha, \partial\beta)$, *and conversity is also true.*

As in the proof of (2.19) $-\gamma$ is represented by a mapping $G:(\dot{I}^{p+q}, 0_*)\to(X, x_0)$ such that $G\,|\,J^{p+q-1}=(f, g)_r$, where f and g are representives of α and β respectively. Also $E(\gamma)$ is represented by a mapping $F':(\dot{I}^{p+q+1}, 0_*)\to(E(X), x_0)$ such that $F'\,|\,J^{p+q}=F$, $F'(I_+^{p+q})\subset\hat{X}_+$ and $F'(I_-^{p+q})\subset\hat{X}_-$. $\chi_1(x)=\bar{\chi}(x, 1)$ gives a homeomorphism $\chi_1:(J^{p+q}; \dot{I}_+^{p+q}, \dot{I}_-^{p+q})\to(K^{p+q}; J^{p+q-1}, J^{p+q-1})$, and there is a homeomorphism $\omega:\dot{I}^{p+q+1}\to\dot{I}^{p+q+1}$ such that $\omega(I^{p+q})=I_+^{p+q}$, $\omega(I_{p+q}^{p+q})=I_-^{p+q}$ and $\omega\,|\,K^{p+q}=\chi_1^{-1}$. It is not so difficult to show that the map ω has degree (-1). Therefore we have that the composite map, $F'\circ\omega:\dot{I}^{p+q+1}\to E(A)$ represents $(-1)\gamma$ hence represents $(-1)\delta$, and that $F\circ\omega(I^{p+q})\subset A_+$, $F\circ\omega(I_{p+q}^{p+q})\subset A_-$ and $F\circ\omega\,|\,K^{p+q}=(\bar{f}, g)_t$. As in the proof of the lemma (2.28) $F\circ\omega$ represents $(-1)^{p(q+1)+1}\gamma'\in\pi_{p+q}(E((A))$, where γ' is represented by the Hopf construction of mapping of type $(\alpha, \partial\beta)$. And the proof of conversity follows from the exactness in the above diagram.

iv) J-homomorphism

Denote the group of the rotations of n-sphere by R_n, and denote the identity by $r_0\in R_n$. Let $f:(\dot{I}^{p+1}, 0_*)\to(R_n, r_0)$ be a representative of $\alpha\in\pi_p(R_n)$, and $\bar{f}:(\dot{I}^{p+1}\times\dot{I}^{n+1}, 0_*)\to(S^n, y_*)$ be a mapping defined by $\bar{f}(x, y)=f(x)(\varepsilon_n(y))$. The homotopy class of the Hopf construction of $\tilde{f}$ is denoted by $J(\alpha)\in\pi_{p+n+1}(S^{n+1})$, which was given by G. W. Whitehead [20] and he showed that the operation J induces homomorphism

$$J:\pi_p(R^n)\to\pi_{p+n+1}(S^{n+1}).$$

The projection $\kappa:R_n\to S^n$ given by $\kappa(x)=x(y_*)$ is the fibre map with the fibre R_{n-1}, so that κ induces isomorphism $\kappa^*:\pi_p(R_n,R_{n-1})\to\pi_p(S^n)$. Let $\iota_n\in\pi_n(S^n)$ be the element represented by $\psi_n:(I^n,\dot{I}^n)\to(S^n,y_*)$. Define a homomorphism $K:\pi_p(R_n,R_{n-1})\to\pi_{p+n+1}(S^{n+1};E_+^{n+1},E_-^{n+1})$ by setting ($p>2$)

$$K(\alpha)=[\partial_-^{-1}(\kappa^*(\alpha)),\partial_+^{-1}\iota_n]_t\quad\text{for }\alpha\in\pi_p(R_n,R_{n-1}),\tag{2.33}$$

where $\partial_-:\pi_{p+1}(E_-^{n+1},S^n)\to\pi_p(S^n)$ and $\partial_+:\pi_{n+1}(E_+^{n+1},S^n)\to\pi_n(S^n)$ are boundary homomorphisms (isomorphisms).

Lemma (2.34) *In the diagram*

$$\begin{array}{ccccccccc}
\cdots\to\pi_p(R_{n-1}) & \xrightarrow{i^*} & \pi_p(R_n) & \xrightarrow{j^*} & \pi_p(R_n,R_{n-1}) & \xrightarrow{\partial} & \pi_{p-1}(R_{n-1})\to\cdots \\
\downarrow J & E & \downarrow J & I & \downarrow K & \Delta & \downarrow J \\
\cdots\to\pi_{p+n}(S^n) & \longrightarrow & \pi_{p+n+1}(S^{n+1}) & \longrightarrow & \pi_{p+n+1}(S^{n+1};E_+^{n+1},E_-^{n+1}) & \longrightarrow & \pi_{p+n+1}(S^n)\to\cdots
\end{array}$$

relations $E\circ J=-J\circ i^*$, $I\circ J=(-1)^{pn+1}K\circ j^*$ *and* $\Delta\circ K=(-1)^{pn}J\circ\partial$ *hold*.

The first relation was proved in previous paper [18], and the second relation follows from (2.28). To show the third relation we ralize the operation K. Let $f:(J^p,\dot{I}^p)\to(R_n,R_{n-1})$ be a mapping such that $f\circ P_p$ is a representative of $\alpha\in\pi_p(R_n,R_{n-1})$, and let $\tilde{f}:J^p\times\dot{I}^{n+1}\to S^n$ be a mapping given by $f(x,y)=f(x)(\varepsilon_n(y))$. Since the element of R_{n-1} is regarded as the element of R_n, which maps hemispheres E_+^n and E_-^n to E_+^n and E_-^n respectively, and cincide to R_{n-1} on S^{n-1}, $\tilde{f}$ maps $J^p\times\dot{I}_+^{n+1}$ and $\dot{I}^p\times\dot{I}_-^{n+1}$ to E_+^n and E_-^n respectively. A mapping $F:(K^{p+n+1};J^{p+n},J_0^{p+n},0_*)\to(S^{n+1};E_+^{n+1},E_-^{n+1},y_*)$ is defined on $J^p\times I^n$ by setting $F\,|\,J^p\times\dot{I}^{q+1}=\tilde{f}$, by extending $F\,|\,\dot{I}^p\times J^n$ over $I^p\times J^n$ such that $F(I^p\times J^n)\subset E_-^{n+1}$, and elsewhere satisfying the condition (2.20). Then $F\circ P'_{p+n+1}$ represents $[\partial_-^{-1}\kappa^*(\alpha),\partial_+^{-1}\beta]_t=K(\alpha)$. Since $F\circ P'_{p+n}|\dot{I}^{p+n}:(\dot{I}^{p+n},0_*)\to(S^n,y^*)$ maps $I^p\times\dot{I}^n$ and $\dot{I}^p\times I^n$ into E_-^{n+1} and E_-^{p+1} respectively, and $F(x,y)=f(x)(\varepsilon_{n-1}(y))$ for $(x,y)\in\dot{I}^p\times\dot{I}^n$, we have from (2.25) that $F\,|\,\dot{I}^{p+n}=F\circ P'_{p+n}|\dot{I}^{p+n}$ represents $J(\partial(\alpha))$, hence we have $\Delta\circ K(\alpha)=(-1)^{pn}J(\partial(\alpha))$.

Chapter 3. A generalization of Hopf and Freudenthal Invariants.

i) Denote a subspace $A\times b_0\cup a_0\times B$ of $A\times B$ by $A\vee B$, and let $i_1:A\to A\vee B$, $i_2:B\to A\vee B$, $p_1:A\times B\to A$, and $p_2:A\times B\to B$ be mappings given by $i_1(a)=(a,b_0)$, $i_2(b)=(a_0,b)$, $p_1(a,b)=a$ and $p_2(a,b)=b$ respectively. It was shown by G. W. Whitehead [22] that the injection homomorphisms $i_1^*:\pi_n(A)\to\pi_n(A\vee B)$ and $i_2^*:\pi_n(B)\to\pi_n(A\vee B)$ and the boundary homomorphism $\partial:\pi_{n+1}(A\times B,A\vee B)\to\pi_n(A\vee B)$ are isomorphisms into, and that there is a direct sum decomposition ($n>1$).

$$\pi_n(A \vee B) = i_1^* \pi_n(A) + i_2^* \pi_n(B) + \partial \pi_{n+1}(A \times B, A \vee B)$$

with projections to these direct factors $p_1^*: \pi_n(A \vee B) \to \pi_n(A)$, $p_2^*: \pi_n(A \vee B) \to \pi_n(B)$ and $Q_0: \pi_n(A \vee B) \to \pi_{n+1}(A \times B, A \vee B)$. For $\alpha \in \pi_n(A \vee B)$, we have $\alpha = i_1^* p_1^*(\alpha) + i_2^* p_2^*(\alpha) + \partial Q_0(\alpha)$.

From the exactness of the following two sequences

$$\cdots \to \pi_n(B) \xrightarrow{i_2^*} \pi_n(A \vee B) \xrightarrow{j} \pi_n(A \vee B, B) \longrightarrow \pi_{n+1}(B) \longrightarrow \pi_{n-1}(A \vee B) \to \cdots$$

and

$$\cdots \to \pi_n(A) \xrightarrow{i \circ i_1^*} \pi_n(A \vee B, B) \xrightarrow{j'} \pi_n(A \vee B\,;\, A, B) \to \pi_n(A) \to \pi_n(A \vee B, B) \to \cdots$$

we see that the composition $j' \circ j \circ \partial: \pi_{n+1}(A \times B, A \vee B) \to \pi_n(A \vee B\,;\, A, B)$ is isomorphism onto for $n \geqq 3$. Define a isomorphism

$$Q: \pi_n(A \vee B\,;\, A, B) \to \pi_{n+1}(A \times B, A \vee B) \tag{3.1}$$

by setting $Q = (j' \circ j \circ \partial)^{-1}$, then $Q_0 = Q \circ j' \circ j$.

Set $S_1^r = S^r \times y_*$, $S_2^r = y_* \times S^r$ and $y_0 = y_* \times y_*$, and consider the following diagram, in which the commutativity relations hold;

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n-1}(S^{r-1}) \arrow[r, "E"] \arrow[d, "\varphi_r^*"] & \pi_n(S^r) \arrow[r, "I"] & \pi_n(S^r\,;\, E_+^r, E_-^r) \arrow[r, "\Delta"] \arrow[d, "\varphi_r^*"] & \pi_{n-2}(S^{r-1}) \arrow[r] & \cdots \\
& \pi_n(S_1^r \vee S_2^r) \arrow[rr, "I'"] \arrow[drr, shift right, "Q_0"'] & & \pi_n(S_1^r \vee S_2^r\,;\, S_1^r, S_2^r) \arrow[d, "Q"] & & \\
& & & \pi_{n+1}(S_1^r \times S_2^r, S_1^r \vee S_2^r) \arrow[r, "\phi_{r,r}^*"] \arrow[ull, shift right, "\partial"'] & \pi_{n+1}(S^{2r}) & \\
& & & \pi_{n+1}(I^{2r}, \dot{I}^{2r}) \arrow[u, "\psi_{r,r}^*"] \arrow[r, "\partial'"] & \pi_n(S^{2r-1}) \arrow[u, "E"'] &
\end{tikzcd}$$

then our *Hopf homomorphisms* $H: \pi_n(S^r, E_+^r, E_-^r) \to \pi_{n+1}(S^{2r})$ and $H_0: \pi_n(S^r) \to \pi_{n+1}(S^{2r})$ are defined by setting ($n \geqq 3$)

$$H = \phi_{r,r}^* \circ Q \circ \varphi_r^* \text{ and } H_0 = H \circ I\,. \tag{3.2}$$

The generalized Hopf homomorphism $H': \pi_n(S^r) \to \pi_n(S^{2r-1})$ of G. W. Whitehead [22] [23] are given by $H' = \partial' \circ \psi_{r,r}^{*-1} \circ Q_0 \circ \varphi_r^*$ for $n \leqq 4r-4$, and from the commutativity of the above diagram we have $H_0 = H' \circ E$. Since E is isomorphic for $n \leqq 4r-4$, we have that H' is *equivalent* to H_0.

As is shown in [9] and [18], we have

$$H_0 \circ E = 0\,, \tag{3.3}$$

$$H_0(\beta \circ E(\alpha)) = H_0(\beta) \circ EE(\alpha)\,. \tag{3.4}$$

Theorem (3.5) *If* $\alpha \in \pi_p(E_-^r, S^{r-1})$, $\beta \in \pi_q(E_+^r, S^{r-1})$, *then*

$$H[\alpha, \beta]_t = (-1)^{q+1} E((\partial\alpha) * (\partial\beta))\,.$$

Proof. Let $f_0: (I^{q-1}, \dot{I}^{q-1}) \to (S^{r-1}, y_*)$ be a representative of $\partial\beta \in \pi_{q-1}(S^{r-1})$, then β is represented by $f(x_1, \dots, x_q) = d_{r-1}(f_0(x_1, \dots, x_{q-1}), x_q)$. By (1.7), $\varphi_r(f(x)) = (Ef_0(x), y_*)$, and we have $\varphi_r^*(\beta) = i_1^* E(\partial\beta)$. Similarly we have

$\varphi_r^*(\alpha)=i_2^*E(\partial\alpha)$. By (3.2), (3.1), (2.12), (2.19) and (2.22) we have

$$\begin{aligned} H[\alpha,\beta]_t &= \phi_{r,r}^*\circ Q\circ\varphi_r^*[\alpha,\beta]_t = \phi_{r,r}^*\circ Q[i_2^*E(\partial\alpha), i_1^*E(\partial\beta)]_t \\ &= (-1)^{q+1}\phi_{r,r}^*(j'\circ j\circ\partial)\circ(j'\circ j\circ\partial)(E(\partial\alpha)\times E(\partial\beta)) \\ &= (-1)^{q+1}\phi_{r,r}^*(E(\partial\alpha)) = (-1)^{q+1}E((\partial\alpha)*(\partial\beta)). \end{aligned}$$

Combining this theorem to lemma (2.28) we have

Corollary (3.6) *If $\gamma\in\pi_{p+q-1}(S^r)$ is represented by the Hopf construction of a mapping: $\dot I^p\times\dot I^q\rightarrow S^{r-1}$ of type (α,β) $(\alpha\in\pi_{p-1}(S^{r-1}),\ \beta\in\pi_{q-1}(S^{r-1}))$, then* $H_0(\gamma)=(-1)^{pq}E(\alpha*\beta)$.

ii) Generalized Freudenthal invariants Λ',Λ'' of G. W. Whitehead are defined on the group π_r^n which is isomorphic to $\pi_n(S^r;E_+^r,E_-^r)$, and have the properties $\Lambda'(\alpha)=(-1)^r\Lambda''(\alpha)$ and $\Lambda'(\alpha)-\Lambda''(\alpha)=(-1)^{r+1}EEH_0(\Delta(\alpha))$. The following theorem due to G. Takeuchi shows that our Hopf invariant H may be used in place of Λ'.

Theorem (3.7) *$H(\alpha)-(-1)^r\iota_{2r}\circ H(\alpha)=(-1)^{r+1}EEH_0(\Delta(\alpha))$ for $\alpha\in\pi_n(S^r;E_+^r,E_-^r)$ $(n\geqq 5,\ r\geqq 2)$.*

To prove the theorem we need several preparations. Let a homomorphism $A:\pi_{n-1}(S_1^{r-1}\times S_2^{r-1},S_1^{r-1}\vee S_2^{r-1})\rightarrow\pi_{n+1}(S_1^r\times S_2^r,S_1^r\vee S_2^r)$ be induced by the formula $Af(x_1,\dots,x_{n+1})=(d_{r-1}(p_1f(x_1,\dots,x_{n-2},x_{n+1}),2x_{n-1}-1),d_{r-1}(p_2f(x_1,\dots,x_{n-2},x_{n+1}),2x_n-1)$ where $f:(I^{n-1},I^{n-2},J^{n-2})\rightarrow(S_1^{r-1}\times S_2^{r-1},S_1^{r-1}\vee S_2^{r-1},y_0)$ is a mapping. As is shown by Hilton [9], in the diagram

$$\begin{array}{ccccc} \pi_{n-1}(S_1^{r-1}\times S_2^{r-1},S_1^{r-1}\vee S_2^{r-1}) & \xrightarrow{A} & \pi_{n+1}(S_1^r\times S_2^r,S_1^r\vee S_2^r) & \underset{\partial}{\overset{Q_0}{\rightleftarrows}} & \pi_n(S_1^r\vee S_2^r) \\ \downarrow{\scriptstyle\phi_{r-1,r-1}^*} & & \downarrow{\scriptstyle\phi_{r,r}^*} & \uparrow\downarrow{\scriptstyle\sigma_r^*} & \\ \pi_{n-1}(S^{2r-2}) & \xrightarrow{E\circ E} & \pi_{n+1}(S^{2r}) & & \pi_{n+1}(S_1^r\times S_2^r,S_1^r\vee S_2^r) \end{array}$$

the relations $\phi_{r,r}^*\circ A=(-1)^rEE\circ\phi_{r-1,r-1}^*$, $\phi_{r,r}^*\circ\sigma_r^*=(-1)^r\iota_{2r}\circ\phi_{r,r}^*$ and $\sigma_r^*\circ Q_0=Q_0\circ\sigma_r^*$ hold. The restriction $F=Af|I^n$ satisfies condition

$$F(I^{n-1}\times\dot I^1\times S^1)\subset S_1^r,\ F(I^{n-2}\times\dot I\times I^2)\subset S_2^r,\ F(I^n\times(1))=y_0$$

$$(3.8)\qquad \begin{aligned} F(x_1,\dots,x_n)&\in S_1^r\vee E_+^r &&\text{if } x_{n-1}\leqq x_n \text{ and } x_{n-1}\geqq 1-x_n,\\ &\in S_1^r\vee E_-^r &&\text{if } x_{n-1}\geqq x_n \text{ and } x_{n-1}\leqq 1-x_n,\\ &\in E_+^r\vee S_2^r &&\text{if } x_{n-1}\geqq x_n \text{ and } x_{n-1}\geqq 1-x_n,\\ &\in E_-^r\vee S_2^r &&\text{if } x_{n-1}\leqq x_n \text{ and } x_{n-1}\leqq 1-x_n, \end{aligned}$$

and $F(x_1,\dots,x_{n-1},1/2,1/2,0)=\partial f(x_1,\dots,x_{n-2})$.

If a mapping $F:\dot I^{n+1}\rightarrow S_1^r\vee S_2^r$ satisfies the condition (3.8), the mapping $\partial f:(I^{n-2},\dot I^{n-2})\rightarrow(S_1^{r-1}\vee S_2^{r-1},y_0)$ represents an element α_0 of $\pi_{n-2}(S_1^{r-1}\vee S_2^{r-1})$.

Since $F(x_1,\dots,x_{n-1},0,0)=(E(p_1\circ\partial f)(x_1,\dots,x_{n-1}),y_0)$, the restriction $F|I^n: I^n\rightarrow S_1^r$ represents a nullhomotopy of $E(p_1\circ f)$, and we haue $E(p_1^*\alpha_0)=0$. Similarly we have $E(p_2^*\alpha_0)=0$. Coversely for any mapping $f:(I^{n-2},\dot I^{n-2})\rightarrow(S_1^{r-1}\vee S_2^{r-1},y_0)$ which satisfiie the condition $E(p_1\circ f)\simeq 0$ and $E(p_2\circ f)\simeq 0$, there is a mapping $F: \dot{I}^{r+1} \to S_1^r \vee S_2^r$ which satisfies the condition (3.8).

Since $S_1^{r-1} \vee S_2^{r-1}$ is contractible in $E_\pm^r \vee E_\pm^r$, we have that if two mappings $f, g: (I^{n-1}, \dot{I}^{n-1}) \to (E_\pm^r \vee E_\pm^r, S^{r-1} \vee S^{r-1})$ coincide on $\dot{I}^{n-1}$, then f is homotopic to g rel. $\dot{I}^{n-1}$. This shows that if two mappings F and F' satisfy the condition (3.8) and homotopic on $I^{n-1} \times (1/2) \times (1/2)$, then F' is homotopic to a mapping F'' (in the homotopy the condition (3.8) holds) such that $F''(x_1, \dots, x_n, 0) = F(x_1, \dots, x_n, 0)$ if $x_{n-1} = x_n$ or $x_{n-1} = 1 - x_n$. It is not so difficult to show that the difference $\{F\} - \{F''\}$ is the sum of four elements, which are represented by mappings of forms: $\dot{I}^{n+1} \to E_+^r \vee S_2^r$, $E_-^r \vee S_2^r$, $S_1^r \vee E_+^r$ and $S_1^r \vee E_2^r$ respectively. Since E_+^r and E_-^r are contractible, $\{F\} - \{F'\}$ is in $i_1^* \pi_n(S^r) + i_2^* \pi_n(S^r) \subset \pi_n(S_1^r \vee S_2^r)$ and therefore $Q_0\{F\} - Q_0\{F'\} = Q_0(\{F\} - \{F'\}) = 0$. And further calculation shows that the correspondence $\{f\} \to Q_0\{F\}$ induces a homomorphism

$$\bar{A}: [\pi_{n-2}(S_1^{r-1} \vee S_2^{r-1})] \to \pi_{n+1}(S_1^{r-1} \times S_2^{r-1}, S_1^{r-1} \vee S_2^{r-1})$$

where $[\pi_{n-2}(S_1^{r-1} \vee S_2^{r-1})]$ is a subgroup of $\pi_{n-2}(S_1^{r-1} \vee S_2^{r-1})$ whose elements satisfy the conditions $E(p_1^* \alpha) = 0$ and $E(p_2^* \alpha) = 0$.

If $g: (I^{n-2}, \dot{I}^{n-2}) \to (S^{r-1}, y_*)$ is a mapping such that $E(\{g\}) = 0$, and let g_t be a nullhomotopy of $g_0 = Eg$, we define a mapping $G: \dot{I}^{n+1} \to S^r$ by $G(x_1, \dots, x_n, 0) = Eg(x_1, \dots, x_{n-1})$, $G(x_1, \dots, x_{n-1}, \pm 1, t) = g_t(x_1, \dots, x_{n-1})$ and $G(J^n - I^{n-2} \times \dot{I}^1 \times I^2) = y_0$, then $i_1 \circ G$ satisfies the condition (3.8). Therefore if $\alpha \in i_1^* \pi_{n-2}(S^{r-1}) \cap [\pi_{n-2}(S_1^{r-1} \vee S_2^{r-1})]$, then we have $\bar{A}(\alpha) = Q i_1^* \{G\} = 0$. Similarly if $\alpha \in i_2^* \pi_{n-2}(S^{r-1}) \cap [\pi_{n-2}(S_1^{r-1} \vee S_2^{r-1})]$, we have $\bar{A}(\alpha) = 0$. If $\alpha \in \pi_{n-1}(S_1^{r-1} \times S_2^{r-1}, S_1^{r-1} \vee S_2^{r-1})$, we have obviously $\bar{A}(\partial \alpha) = A(\alpha)$, hence if $\alpha \in [\pi_{n-2}(S_1^{r-1} \vee S_2^{r-1})]$ we have $\bar{A}(\alpha) = \bar{A}(i_1^r p_1^*(\alpha) + i_2^* p_2^*(\alpha) + \partial Q_0(\alpha)) = A(Q_0(\alpha))$. Consequently we have

Lemma (3.9) *if a mapping F satisfies the condition* (3.8), *and if $F | I^{n-2} \times (1/2) \times (1/2)$ represents $\alpha \in \pi_{n-2}(S_1^{r-1} \vee S_2^{r-1})$, then we have $Q_0\{F\} = A \circ Q_0(\alpha)$.*

Proof of theorem (3.7). Let $f: (I^n; I_+^{n-1}, I_-^{n-1}, J^{n-1}) \to (S_1^r; E_+^r, E_-^r, y_*)$ be a representative of $\alpha \in \pi_n(S^r; E_+^r, E_-^r)$, and let $\Delta f: (I^{n-2}, \dot{I}^{n-2}) \to (S^r, y_*)$ be representative of $\Delta\alpha$ such that $\Delta f(x_1, \dots, x_{n-2}) = f(x_1, \dots, x_{n-2}, 1/2, 0)$. Since $f = | I^{n-1}$ is homotopic to $E\Delta f$, we may assume that $f | I^{n-1} = E\Delta f$. Set $F = \varphi_r \circ f$ and define a mapping $F': (I^n; I_+^{n-1}, I_-^{n-1}, J^{n-1}) \to (S_1^r \vee S_2^r; S_2^r, S_1^r, y_0)$ by setting

$$\begin{aligned} F'(x_1, \dots, x_n) &= F(x_1, \dots, x_{n-1}, 2x_n - 1) && 1/2 \leqq x_n \leqq 1, \\ &= \varphi_r \circ \rho_r(2\pi x_n) E\Delta f(x_1, \dots, x_{n-1}) && 0 \leqq x_n \leqq 1/2. \end{aligned}$$

It is easily verified that F' is homotopic to a mapping $\varphi_r \circ \rho_r(\pi) \circ F = \sigma_r \circ \varphi_r \circ F$. Since the homomorphism $I': \pi_n(S_1^r \vee S_2^r) \to \pi_n(S_1^r \vee S_2^r; S_1^r, S_2^r)$ is onto there is a mapping $\bar{F}: \dot{I}^{n+1} \to S_1^r \vee S_2^r$ such that $\bar{F} | I^n = F$, $\bar{F}(I_+^{n-1} \times (0) \times I^1) \subset S_1^r$, $\bar{F}(I_-^{n-1} \times (0) \times I^1) \subset S_1^r$, and $\bar{F}(K^r) = y_0$, and we have $I'\{\bar{F}\} = \varphi_r^*(\alpha)$. And also there is a mapping $\bar{F}': \dot{I}^{n+1} \to S_1^r \vee S_2^r$ such that $\bar{F}' | I^n = F'$, $\bar{F}'(I_-^{n-1} \times (0) \times I^1) \subset S_2^r$, $\bar{F}'(I_-^{n-1} \times (0) \times I^1) \subset S_1^r$, $\bar{F}'(K^n) = y_0$ and $I'(\{\bar{F}'\}) = \sigma_r^* \circ \varphi_r^*(\alpha)$. The difference $\{\bar{F}\} - \{\bar{F}'\}$ is represented by a mapping $F_0: \dot{I}^{n+1} \to S_1^r \vee S_2^r$ such that $F_0(I^n \times (1) \cup \dot{I}^{n-1} \times I^2) = y_0$, $F_0(I_+^{n-1} \times (0) \times I^1 \cup I_-^{n-1} \times (1) \times I^1) \subset S_1^r$, $F_0(I_-^{n-1} \times (0) \times I^1 \cup I_+^{n-1} \times (1) \times I^1) \subset S_2^r$ and $F_0(x_1, \dots, x_n) = \varphi_r \circ \rho_r(-\pi x_n) \circ E\Delta f(x_1, \dots, x_{n-1})$.

Define a mapping $\omega: \dot{I}^{n+1} \to \dot{I}^{n+1}$ of degree -1 by setting $\omega((x_1, \dots, x_{n+1}) = (x_1, \dots, x_{n-2}, \omega'(x_{n-1}, x_n), x_{n+1})$ where $\omega'(x, y) = (1-2(1-x)(1-y), 1-2(1-x)y)$ for $1/2 \leqq x \leqq 1$ and $\omega'(x, y) = (2xy, 2x(1-y))$ for $0 \leqq x \leqq 1/2$. Since ω is homeomorphic on $\dot{I}^{n+1} - I^{n-2} \times \dot{I}^1 \times I^2$ and F_0 maps $I^{n-1} \times \dot{I}^1 \times I^2$ into the single point y_0, there is a mapping $\bar{F}_0: \dot{I}^{n+1} \to S_1^r \vee S_2^r$ such that $\bar{F}_0 \circ \omega = F_0$. It is verified from (1.4)$'$ and (1.7) that $\bar{F}_0$ satisfies the conditions (3.8), and from (3.9) we have

$$\begin{aligned}
&H(\alpha) - (-1)^r \iota_{2r} \circ H(\alpha) = \phi_{r,r}^* \circ \sigma_r^* \circ Q \circ \varphi_r^*(\alpha) \\
&= \phi^* \circ Q \circ I'\{\bar{F}\} - \phi^* \circ Q \circ I'\{\bar{F}'\} = \phi^* \circ Q \circ (\{F\} - \{\bar{F}\}) \\
&= \phi^* \circ Q \circ \{\bar{F}_0\} = -\phi^* \circ Q\{F_0\} = -\phi^* \circ A \circ Q \circ \left\{\varphi_r^* \circ \rho_r\left(\frac{\pi}{2}\right) \circ \Delta f\right\} \\
&= -\phi^* \circ A \circ Q \circ \varphi_{r-1}^*(\Delta\alpha) = (-1)^{r+1} EEH_0(\Delta\alpha),
\end{aligned}$$

and the proof of the theorem (3.9) is accomplished.

Since $\Delta \circ I = 0$, we have

Corollary (3.10) $\qquad H_0(\alpha) = (-1)^r \iota_{2r} \circ H_0(\alpha)$.

If $\alpha \in \pi_p(S^r)$ and $\beta \in \pi_q(S^r)$, there are elements $\bar{\alpha} \in \pi_{p+1}(E_-^{r+1}, S^r)$ and $\bar{\beta} \in \pi_{q+1}(E_+^{r+1}, S^r)$ such that $\partial\bar{\alpha} = \alpha$, $\partial\bar{\beta} = \beta$ and $\Delta[\bar{\alpha}, \bar{\beta}]_t = [\alpha, \beta]$. By (3.5), (3.7) and (2.4), we have $(-1)^r EEH_0[\alpha, \beta] = H[\bar{\alpha}, \bar{\beta}]_t - (-1)^{r+1} \iota_{2r+2} \circ H[\bar{\alpha}, \bar{\beta}]_t = (-1)^q E(\alpha * \beta) - (-1)^{r+1} \iota_{r+2} \circ (-1)^q E(\alpha * \beta) = (-1)^q (1 - (-1)^{r+1}) E(\alpha * \beta)$, and therefore

Corollary (3.11) $\quad EEH_0[\alpha, \beta] = 2(-1)^q E(\alpha * \beta) \quad$ *if r is even,*
$\qquad\qquad\qquad\qquad\qquad\qquad = 0 \qquad\qquad\qquad$ *if r is odd.*

iii) Next we shall define a Hopf invariant to more general group $\pi_p(X^*; \mathcal{E}^n, X)$. Let $\varphi_i; (X^*; \mathcal{E}^n, \tilde{X}) \to (X \vee S^n; S^n, \tilde{X})$ be a mapping identifying the subset $\bigcup_{i \neq j} \mathcal{E}_j^n \cup \dot{\mathcal{E}}_i^n$ to the single point $x_0 = S^n \frown \tilde{X}$, and let $\phi_n; (\tilde{X} \times S^n, \tilde{X} \vee S^n) \to (E^n(X), x_0)$ be the shrinking map in (2.8). Then a *Hopf homomorphism* $H = \pi_p(X^*; \mathcal{E}^n, X) \to \pi_{p+1}(E^n(X))$ is defined by setting $H = \phi_n^* \circ Q \circ \varphi_i^*: \pi_p(X^*; \mathcal{E}^n, X) \to \pi_p(\tilde{X} \vee S^n; S^n, \tilde{X}) \to \pi_{p+1}(\tilde{X} \times S^n, \tilde{X} \vee S^n) \to \pi_{p+1}(E^n(X))$. Define a homomorphism P_i: $\pi_{p-n+1}(X, \dot{\mathcal{E}}^n) \to \pi_p(X^*; \mathcal{E}^n, X)$ by setting $P_i(\alpha) = [\alpha, \iota]$, where ι is a generator of $\pi_n(\mathcal{E}_i^n, \dot{\mathcal{E}}_i^n)$. By (3.1), (2.12), (2.19) and (2.10) we have $H_i P_i(\alpha) = \phi_n^* \circ Q \circ \varphi_i^* [\alpha, \iota_i]_t = \phi_n^* \circ Q \circ [\varphi_i^*(\alpha), \iota_n]_t = (-1)^n \phi_n^* (j' \circ j \circ \partial)^{-1} j' \circ j \circ \partial (\varphi_i^*(\alpha) \times \iota_n) = (-1)^q \phi_n^* (\varphi_i^*(\alpha) \times \iota_n) = (-1)^n E^n(\varphi_i^*(\alpha))$. If $i \neq j$, $\varphi_i^*[\alpha, \iota_j] = 0$ and hence $H_i P_j = 0$.

$$(3.12) \qquad H_i P_j(\alpha) = (-1)^n E\varphi_i^*(\alpha) \qquad i = j, \\ \qquad\qquad = 0 \qquad\qquad\qquad i \neq j.$$

If $(X, \dot{\mathcal{E}}^n)$ is smooth and m-connected, and if $\dot{\mathcal{E}}^n$ is r-connected, then $\varphi_i^*: \pi_{p-n+1}(X, \dot{\mathcal{E}}^n) \to \pi_{p-n+1}(\tilde{X})$ is isomorphism onto for $p - n + 1 \leqq m + p$ by (1.26), and $E^n: \pi_{p-n+1}(\tilde{X}) \to \pi_{p+1}(E^n(\tilde{X}))$ is isomorphism onto for $p - n + 1 \leqq 2m$ by (2.6). Then the following theorem is algebraic consequence of the above considerations:

Theorem (3.13) *with above hypotheses $\pi_p(X^*; \mathcal{E}^n, X)$ has a direct factor isomorphic to $\pi_{p-n+1}(X, \dot{\mathcal{E}}^n)\otimes\pi_n(\mathcal{E}^n, \dot{\mathcal{E}}^n)$ for $p\leqq n+m+min(m,r)-1$.*

Combining this theorem to (1.27) we have

Corollary (3.14) *$\pi_p(\mathcal{E}^n, \dot{\mathcal{E}}^n)$ has a direct factor isomorphic to $\sum_i \pi_p(E^n, S^{n-1})\oplus \pi_{p-n+1}(\dot{\mathcal{E}}^n)\otimes\pi_n(\mathcal{E}^n, \dot{\mathcal{E}}^n)$ for $p\geqq n+min.(2r, n-2)-1$, where tensor product $\otimes$ is induced by the relative product.*

Chapter 4. Some elements of $\pi_n(S^r)$.

i) It is well known that the mapping $\psi_n: (I^n, \dot{I}^n)\rightarrow(S^n, y_*)$ represents a generator ι_n of the infinite cyclic group $\pi_n(S^n)\approx Z$, and that $\pi_n(S^1)=0$ for $n>1$, and $\pi_n(S^r)=0$ for $n<r$.

There is fibre mappings $h_r: S^{2r-1}\rightarrow S^r(r=2,4,8)$ with fibre S^{r-1}, and they are represented by the Hopf construction of mappings of type $(\iota_{r-1}, \iota_{r-1})$. If h'_r is another fibre mapping, then there is a mapping $\chi: S^{2r-1}\rightarrow S^{2r-1}$ of degree 1 such that $h_r=h'_r\circ\chi$, and therefore $H(\{h_r\})=E(\ _{r-1}*\iota_{r-1})=\iota_{2r}=\pm H(\{h_r'\})$. As is shown in [5], the homomorphisms $h_r^*: \pi_n(S^{2r-1})\rightarrow\pi_n(S^r)$ and $E; \pi_{n-1}(S^{r-1})\rightarrow \pi_n(S^r)$ are isomorphisms into and

$$(4.1)\qquad \pi_n(S^r) = h_r^*\pi_n(S^{2r-1})\oplus E(\pi_{n-1}(S^{r-1})).$$

ii) By (4.1) $\pi_3(S^2)\approx\pi_3(S^3)\approx Z$ and its generator η_2 is represented by h_2. The fact $\pi_{n+1}(S^n)\approx Z_2$ for $n\geqq 3$ is shown by H. Freudenthal [8], and its generator η_n is the $(n$-$2)$-fold suspension of η_2. It was shown by G. W. Whitehead [23] that $\pi_{n+2}(S^n)\approx Z_2(n\geqq 2)$ and its generator is $\eta_n\circ\eta_{n+1}$.

For convenience we modify the theorem (3.7). Let $\alpha\in\pi_n(S^r)$ be an element such that $E(\alpha)=0$, then there is an element γ of $\pi_{n+2}(S^{r+1}; E_+^{r+1}, E_-^{r+1})$ such that $\Delta(\gamma)=\alpha$. By (3.7) we have $H(\gamma)-(-1)^{r+1}\iota_{2r+2}\circ H(\gamma)=(-1)^r EEH_0(\alpha)$, hence we have $EEH_0(\alpha)=0$ if r is odd. If $n+2\leqq 2(2r+1)-1$, the suspension homomorphism $E: \pi_{n+2}(S^{2r+1})\rightarrow\pi_{n+3}(S^{2r+2})$ is onto and there is an element β of $\pi_{n+2}(S^{2r+1})$ such that $E(\beta)=H(\gamma)$, and therefore we have by (2.4) $(-1)^r EEH_0(\alpha)$ $=E(\beta)-(-1)^{r+1}\iota_{3r+2}\circ E(\beta)=(1-(-1)^{r+1})E(\beta)$. Consequently we have

(4.2) *if $\alpha\in\pi_n(S^r)$ and $E(\alpha)=0$, we have*
$EEH_0(\alpha)=0$ *if r is odd,*
$EEH_0(\alpha)\in 2\pi_{n+3}(S^{2r+2})$ *if r is even and $n\leqq 4r-1$.*

Since $2\pi_{r+2}(S^{r+1})=2\pi_{r+2}(S^r)=0$ for $r\geqq 2$, we have

(4.2)′ *if $\alpha\in\pi_{2r}(S^r)$ or $\alpha\in\pi_{2r+1}(S^r)$ and $r\geqq 2$, and if $H(\alpha)\neq 0$, then $E(\alpha)\neq 0$.*

For example $\eta_3\circ\eta_4\circ\eta_5$ is a nonzero element of $\pi_6(S^3)$.

iii) Let $q=x_1+ix_2+jx_3+kx_4$ be a quaternion, then we may regard a point (x_1, x_2, x_3, x_4) of S^3 as a quaternion of unit absolute value and regard a point (x_1, x_2, x_3) of S^2 as a pure quaternion $ix_1+jx_2+kx_3$ of unit absolute value. The product $p\cdot i\cdot p^{-1}=h(p)$ $(p\in S^3, i=(0,1,0,0))$ defines a fibre mapping $h: S^3\to S^2$ and h represents a generator of $\pi_3(S^2)$.

The products $p\cdot q=f(p,q)$ and $p\cdot q_0\cdot p^{-1}=g(p,q)$ $(p, q\in S^3, q_0\in S^2)$ define mappings $f: S^3\times S^3\to S^3$ and $g: S^3\times S^2\to S^2$ of types (ι_3,ι_3) and $(\pm\eta_2,\iota_2)$ respectively. The hopf construction of f is a fibre mapping, and let its class be $\nu_4\in\pi_7(S_4)$, then $H(\nu_4)=\iota_8$ by (3.6). Let $\alpha_3\in\pi_6(S^3)$ be the class of the Hopf construction of g, the $H(\alpha_3)=\ _6$. Let ν_n and α_n be $(n-4)$- and $(n-3)$-fold suspensions of ν_4 and α_3 respectively. The author proved that [18]

Lemma (4.3) i) *the* $(n-3)$*-fold suspension* $E^{n-3}: \pi_6(S^3)\to\pi_{n+3}(S^n)$ *is isomorphism into for* $n\geqq 5$, *and* $\pi_{n+3}(S^n)/E^{n-3}(\pi_6(S^3))\approx Z_2$,

ii) $[\iota_4,\iota_4]=2\nu_4-\alpha_4$, $2\nu_n=\alpha_n$ *for* $n\geqq 5$ *and* $\eta_n\circ\eta_{n+1}\circ\eta_{n+2}\neq 0$ *for* $n\geqq 2$.

iv) Let f, g and h be mappings as in iii), then a diagram

$$\begin{array}{ccc} S^3\times S^3 & \xrightarrow{f} & S^3 \\ \downarrow{\scriptstyle (i_3\times h)} & & \downarrow{\scriptstyle h} \\ S^3\times S^2 & \xrightarrow{g} & S^2 \end{array}$$

commutes, where $i_3: S^3\to S^3$ is the identity map and $(i_3\times h)(x,y)=(x,h(y))$. The difinitions of Hopf construction, join and suspension shows that $E(\pm\eta_2)\circ\nu_4=\alpha_3\circ(\iota_3*\eta_2)=\alpha_3\circ\eta_6$. By (2.23), (2.24) and (4.3) we have $[\eta_4,\iota_4]=[\iota_4,\iota_4]\circ(\eta_3*\iota_3)=(2\nu_4-\alpha_4)\circ\eta_6=(2\nu_4\circ\eta_6)+\alpha_3\circ\eta_6=\alpha_3\circ\eta_6$, and by (2.27) $\eta_5\circ\nu_6=\alpha_5\circ\eta_8=E[\eta_4,\iota_4]=0$. By (3.4) we have $H(\eta_3\circ\nu_4)=H(\alpha_3\circ\eta_6)=H(\alpha_3)\circ\eta_7=\eta_6\circ\eta_7\neq 0$ hence $\eta_3\circ\nu_4\neq 0$, and by (4.2)′ we have $E(\eta_3\circ\nu_4)=\eta_4\circ\nu_5\neq 0$. Consequently we obtain

Lemma (4.4) $\quad \eta_n\circ\nu_{n+3}=\alpha_n\circ\eta_{n+3}\neq 0\quad$ *for* $n=3$ *and* 4,
$\qquad\qquad =0\quad$ *for* $n\geqq 5$.

v) It was shown in [19] [17] that the homotopy group $\pi_4(R_4)$ of the rotation group R_4 is the cyclic group of order 2 and the boundary homomorphism $\partial: \pi_5(R_5,R_4)\to\pi_4(R_4)$ is onto, and that the generator of $\pi_4(R_4)$ is given by $i^*(\alpha)\circ\eta_3$, where $\alpha\in\pi_3(R_3)$ is represented by $f(p)(q)=p\cdot q(p, q\in S^3)$. From (2.34) we have that $J(i^*(\alpha)\circ\eta_3)=J(i^*(\alpha\circ\eta_3))=-EJ(\alpha\circ\eta_3)=EJ(\alpha\circ\eta_3)$ is represented by product $[\iota_5,\iota_5]$, and $J(\alpha\circ\eta_3)=J(\alpha)\circ(\eta_3*\iota_3)=\nu_4\circ\eta_7$. Therefore $[\ _5,\iota_5]=E(\nu_4\circ\eta_7)=\nu_5\circ\eta_8$, and $\nu_6\circ\eta_9=E[\iota_5,\iota_5]=0$. By (3.4) $H(\nu_4\circ\eta_7)=\eta_8\neq 0$ and $\nu_4\circ\eta_7\neq 0$, and by (4.3)′ $E(\nu_4\circ\eta_7)=\nu_5\circ\eta_8\neq 0$.

Lemma (4.5) $\quad \nu_n\circ\eta_{n+3}\neq 0\quad$ *for* $n=4$ *and* 5
$\qquad\qquad =0\quad$ *for* $n\geqq 6$.

vi) Let q_i be a quaternion, then we may represents a point of C^{4n} by $(q_1,\dots,q_n)$. The equivalence relation $\{(q_1,\dots,q_n)\}=\{(pq_1,\dots,pq_n)\}$ induces quarternion projective space Q^{4n-4} with respect to the indentification mapping q'_{n-1}: $C^{4n}-0_*\to Q^{4n-4}$. Obviously $q'_{n-1}|C^{4n-4}-0_*=q'_{n-2}$. With normalization process we obtain a fibre mapping $q_{n-1}: S^{4n-1}\to Q^{4n-4}$ and its fibre is S^3. The correspondence $(q_1, \dots, q_n) \to \{(q_1, \dots, q_{n'})\} \in Q^{4n}$ gives a homeomorphism: $C^{4n} \to Q^{4n} - Q^{4n-4}$, and this shows that there is a character mapping $\tilde{q}_{n-1}: (E^{4n}, S^{4n-1}) \to (Q^{4n}, Q^{4n-4})$ such that $q_{n-1} | S^{4n-1} = q_{n-1}$. Then we obtain a cell decomposition $Q^{4n} = S^4 \cup e^8 \cup \dots \cup e^{4n}$ of Q^{4n}. The fibre mapping $q_2: (S^{11}, S^7) \to (Q^8, S^3)$ induces isomorphism $q_2^*: \pi_n(S^{11}, S^7) \to \pi_n(Q^8, S^4)$. Consider the diagram

$$\begin{array}{ccccc}
\pi_{11}(S^{11}, S^7) & \xrightarrow{\partial} & \pi_{10}(S^7) & \longrightarrow & \pi_{10}(S^{11}) \\
\downarrow q_2^* & \partial' & \downarrow q_1^* & & \\
\pi_{11}(Q^8, S^4) & \longrightarrow & \pi_{10}(S^4). & &
\end{array}$$

Since S^7 is contractible in S^{11}, ∂ is onto. Let $\tilde{\iota}_8 \in \pi_8(Q^8, S)$ be the class of $\tilde{q}_1$, then $\partial \tilde{\ }_8 = \{q_1\} = \nu_4$. There is an element γ of $\pi_{11}(S^{11}, S^7)$ such that $q_2^*(\gamma) = [\iota_4, \tilde{\iota}_8]_r$, for q_2^* is onto, and therefore $[\iota_4, \nu_4] = \partial[\iota_4, \iota_8]_r = \partial q_2^*(\gamma) = q_1^* \circ \partial'(\gamma)$. Set $\partial'(\gamma) = \beta \in \pi_{10}(S^7)$, then $[\iota_4, \nu_4] = \nu_4 \circ \beta$. We have $H_0[\iota_4, \nu_4] = 2E^{-1}(\iota_4 * \nu_4) = 2\nu_8$ by (3.11), and $H_0(\nu_4 \circ \beta) = E(\beta)$ by (3.4). Since $E: \pi_{10}(S^7) \to \pi_{11}(S^8)$ is isomorphism onto, we have $\beta = 2\nu_7$ and

Lemma (4.6) $[\iota_4, \nu_4] = 2\nu_4 \circ \nu_7$ *and* $2\nu_n \circ \nu_{n+3} = 0$ *for* $n \geqq 5$.

Chapter 5. A construction of mapping.

i) Consider a mapping $F: (I^{n+1}, \dot{I}^{n+1}) \to (X, x_0)$ which satisfies the conditions $(n \geqq 2)$:

(A_1) $F(x_1, \dots, x_{n+1}) = F(x_1, \dots, x_{n-2}, x_{n-1} + 1/2, x_n, x_{n+1})$
for $0 \leqq x_{n-1} \leqq 1/2$ *and* $0 \leqq x_{n+1} \leqq 1/2$,

(A_2) $F(x_1, \dots, x_{n+1}) = F(x_1, \dots, x_{n-1}, x_n + 1/2, x_{n+1})$
for $0 \leqq x_n \leqq 1/2$ *and* $1/2 \leqq x_{n+1} \leqq 1$.

The formula

(5.1) $$f(x_1, \dots, x_n) = F(x_1, \dots, x_{n-2}, x_{n-1}/2, x_n/2, 1/2)$$

represents a map $f: (I^n, \dot{I}^n) \to (X, x_0)$. From (A_1) a map $F_1: (I^n, \dot{I}^n) \to (X, x)$ given by $F_1(x_1, \dots, x_n) = F(x_1, \dots, 2x_{n-1}, x_n, 1/2)$ is the sum $f+f$ on the x_{n-1}-axis, and also from (A_2) the formula $F_0'(x_1, \dots, x_n) = F(x_1, \dots, x_{n-1}, 2x_n, 1/2)$ gives the sum $f+f$ on the x_n-axis. By setting $F_t(x_1, \dots, x_n) = F(x_1, \dots, x_{n-2}, 2x_{n-1}, x_n, t/2)$ and $F_t'(x_1, \dots, x_n) = F_t(x_1, \dots, x_{n-2}, x_{n-1}, 2x_n, (t+1)/2)$, we obtain null-homotopies of F_1 and F_0' respectively. Therefore

(A_3) *f represents an element α of ${}_2[\pi_n(X)]$*,

where ${}_2[\pi_n(X)]$ is the subgroup of $\pi_n(X)$ generated by the elements of order 2.

Conversely for any element α of ${}_2[\pi_n(X)]$, there exists a map $F: (I^{n+1}, \dot{I}^{n+1}) \to (X, x_0)$ satisfying the conditions (A_1), (A_2) and (A_3). Let F and F' be two maps which satisfy the above three conditions, and let f and f' be the restricted maps as in (5.1). Since f and f' represent the same element α, there is a homotopy $f_t: (I^n, \dot{I}^n) \to (X, x_0)$ from $f = f_0$ to $f' = f_1$. Define a homotopy $g_t: (I^n, \dot{I}^n) \to (X, x_0)$ by a rule

$$f_t(x_1,\dots,x_n) = g_t(x_1,\dots,x_{n-2},x_{n-1}/2,x_n/2) = g_t(x_1,\dots,x_{n-2},x_{n-1}/2,(x_n+1)/2)$$
$$= g_t(x_1,\dots,x_n,(x_{n-1}+1)/2,x_n/2) = g_t(x_1,\dots,x_{n-2},(x_{n-1}+1)/2,(x_n+1)/2),$$

then we have $g_0(x_1,\dots,x)=F(x_1,\dots,x_n,1/2)$ and $g_1(x_1,\dots,x_n)=F'(x_1,\dots,x_n,1/2)$. Define two maps F_+ and $F_-:(I^{n+1},\dot{I}^{n+1})\to(X,x_0)$ by setting

$$\begin{aligned} F_+(x_1,\dots,x_n,t) &= F(x_1,\dots,x_n,(2-3t)/2) && 0\leqq t\leqq 1/3,\\ &= g_{3t-1}(x_1,\dots,x_n) && 1/3\leqq t\leqq 2/3,\\ &= F'(x_1,\dots,x_n,(3t-1)/2) && 2/3\leqq t\leqq 1. \end{aligned}$$

and

$$\begin{aligned} F_-(x_1,\dots,x_n,t) &= F'(x_1,\dots,x_n,3t/2) && 0\leqq t\leqq 1/3,\\ &= g_{2-3t}(x_1,\dots,x_n) && 1/3\leqq t\leqq 2/3,\\ &= F(x_1,\dots,x_n,(3-3t)/2) && 2/3\leqq t\leqq 1. \end{aligned}$$

It is easily verified that the sum $F_++(F+F_-)$ on the x_{n+1} -axis is homotopic to F'. Since $F_+(x_1,\dots,x_{n-1},x_n,x_{n+1})=F_+(x_1,\dots,x_{n-1},x_n+1/2,x_{n+1})$ for $0\leqq x_n\leqq 1/2$, F_+ is the sum $F'_++F'_+$ on the x_n -axis, where $F'_+(x_1,\dots,x_{n+1})=F_+(x_1,\dots,x_{n-1},x_n/2,x_{n+1})$, and hence the class of F_+ belongs to $2(\pi_{n+1}(X))$. Similarly the class of F_- belongs to $2(\pi_{n+1}(X))$. Consequently we have

(5.2) *the class $\{F\}$ of in $\pi_{n+1}(X)/2\pi_{n+1}(X)$ depends only on α, and it is denoted by $T(\alpha)$.*

If $n\geqq 3$, and if F_1 and F_2 are representatives of $T(\alpha)$ and $T(\beta)$ respectively, a representative F of $T(\alpha+\beta)$ is given by the sum F_1+F_2 on the x_1 -axis. Therefore $T(\alpha)+T(\beta)=T(\alpha+\beta)$, and we obtain a homomorphism

(5.3) $$T_2: {}_2[\pi_n(X)]\to\pi_{n+1}(X)/2\pi_{n+1}(X) \qquad (n\geqq 3).$$

In the case $n=2$, by theorem (5.15) of [22] and the following theorem we have $T(\alpha+\beta)=T(\alpha)+T(\beta)+\{[\alpha,\beta]\}$.

Theorem (5.4) *$T(\alpha)$ is the class of $\alpha\circ\eta_n$.*

To prove the theorem we shall give a representative of $T(\alpha)$ by spherical mappings. Let $\varepsilon:(E^n,S^{n-1})\to(I^n,\dot{I}^n)$ be the homeomorphism given by

$$\varepsilon(x_1,\dots,x_n)=\left(\frac{1+\rho x_1}{2},\dots,\frac{1+\rho x_n}{2}\right),\ \text{where } \rho=\frac{\sqrt{x_1^2+\cdots+x_2^n}}{\max(|x_1|,\dots,|x_n|)}$$

Define a map $\chi_1: I^{n+1}\to E^n$ by setting for $0\leqq x_{n-1}\leqq 1/2$ and $0\leqq x_n\leqq 1/2$

$$\begin{aligned} \chi_1(x_1,\dots,x_{n+1}) &= \varepsilon^{-1}(x_1,\dots,x_{n-2},2x_n,2x_{n+1}) && \text{if } 0\leqq x_{n+1}\leqq x_{n-1},\\ &= \varepsilon^{-1}(x_1,\dots,x_{n-2},2x_n,2x_{n-1}) && \text{if } x_{n-1}\leqq x_{n+1}\leqq 1-x_n,\\ &= \varepsilon^{-1}(x_1,\dots,x_{n-2},2-2x_{n+1},2x_{n-1}) && \text{if } 1-x_n\leqq x_{n+1}\leqq 1, \end{aligned}$$

and by adjoining the condition

$$\chi_1(x_1,\dots,x_{n+1})=\chi_1(x_1,\dots,x_{n-1},1-x_n,x_{n+1})=\chi_1(x_1,\dots,x_{n-1},1-x_n,x_{n+1}).$$

for the other values of x_{n-1} and x_n

Set $L_+=\{(x_1,\dots,x_{n+1})\in I^{n+1}\mid x_n=1/2, x_{n+1}\geqq 1/2\}$ and $L_-=\{(x_1,\dots,x_{n+1})\in I^{n+1}\mid x_{n-1}=1/2, x_{n+1}\leqq 1/2\}$. We may represent a point (x_1,x_2) of S^1 by a radian θ such that $(\cos\theta,\sin\theta)=(x_1,x_2)$. Define a mapping: $\chi_2: I^{n+1}-\dot{I}^{n+1}-L_+-L_-\to S^1$ by setting

$$\begin{aligned}
\chi_2(x_1,\dots,x_{n+1}) &= -\pi/4 && 0<x_{n-1}<1/2, 1/2<x_n<1 \text{ and } x_{n+1}=1/2,\\
&= \frac{1}{2}\, Arctan \frac{1-2x_n}{1-2x_{n+1}} && 0<x_{n-1}<1/2 \text{ and } 0<x_{n+1}<1/2,\\
&= \frac{\pi}{4} && 0<x_{n-1}<1/2, 0<x_n<1/2 \text{ and } x_{n+1}=1/2,\\
&= \frac{\pi}{2}+\frac{1}{2}\, Arctan \frac{1-2x_{n-1}}{1-2x_{n+1}} && 0<x_n<1/2 \text{ and } 1/2<x_{n+1}<1,\\
&= 3\pi/4 && 1/2<x_{n-1}<1,\ 0<x_n<1/2 \text{ and } x_{n+1}=1/2,\\
&= \pi-1/2\ Arctan \frac{1-2x_n}{1-2x_{n+1}} && 1/2<x_{n+1}<1 \text{ and } 0<x_{n+1}<1/2,\\
&= 5\pi/4 && 1/2<x_{n-1}<1,\ 1/2<x_n<1 \text{ and } x_{n+1}=1/2,\\
&= 3/2\pi-1/2\ Arctan \frac{1-2x_{n-1}}{1-2x_{n+1}} && 1/2<x_n<1 \text{ and } 1/2<x_{n+1}<1.
\end{aligned}$$

For a fixed point $\theta\in S^1$, the inverse image $\chi_2^{-1}(\theta)$ is an open n-cube in I^{n+1} and χ_1 maps $\chi_2^{-1}(\theta)$ homeomorphically onto E^n-S^{n-1}. Therefore the formula $\chi(x)=(\chi_1(x),\chi_2(x))$ gives a homeomorphism $\chi: I^{n+1}-\dot{I}^{n+1}-L_+-L_-\to(E^n-S^{n-1})\times S^1$.

Define a map $\phi: E^n\times S^1\to S^{n+1}$ by $\phi(x_1,\dots,x_n,y_1,y_2)=(x_1,\dots,x_n,\mu y_1,\mu y_2)$, where $(x_1,\dots,x_n)\in E^n$, $(y_1,y_2)\in S^1$ and $\mu=(1-x_1^2-\cdots-x_n^2)^{\frac{1}{2}}$, then ϕ maps $(E^n-S^{n-1})\times S^1$ homeomorphically onto $S^{n+1}-S^{n-1}$.

Define a map $\psi: (I^{n+1}, \dot{I}^{n+1}\cup L_+\cup L_-)\to(S^{n+1},S^{n-1})$ by setting

$$\psi(x)=\phi(\chi(x)) \qquad \text{for } x\in I^{n+1}-\dot{I}^{n+1}-L_+-L_-,$$

and

$$\psi(x)=(\chi_1(x),0,0) \qquad \text{for } x\in \dot{I}^{n+1}\cup L_+\cup L_-,$$

then ψ maps $I^{n+1}-\dot{I}^{n+1}-L_+-L_-$ homeomorphically onto $S^{n+1}-S^{n-1}$.

Since F maps $\dot{I}^{n+1}\cup L_+\cup L_-$ into the single point x_0, there is a unique map $H:(S^{n+1},S^{n-1})\to(X,x_0)$ such that $F=H\circ\psi$. It is verified from the definition of ψ that the map H satisfies the conditions:

(B_1) $H(\phi(x_1,\dots,x_n,\theta))=H(\phi(x_1,\dots,x_{n-1},x_n,\pi-\theta))$ $\quad -\pi/4\leqq\theta\leqq\pi/4$,

(B_2) $H(\phi(x_1,\dots,x_n,\theta))=H(\phi(x_1,\dots,x_{n-1},-x_n,2\pi-\theta))$ $\quad \pi/4\leqq\theta\leqq 3\pi/4$,

(B_3) *and a map* $h:(E^n,S^{n-1})\to(X,x_0)$ *giving by* $h(x_1,\dots,x_n)=H(\phi(x_1,\dots,x_n,y_*))$ ***represents*** α.

Conversely, for any map $H:(S^{n+1},S^{n-1})\to(X,x_0)$ satisfying the above three conditions, the composite map $H\circ\psi=F:(I^{n+1},\dot{I}^{n+1})\to(X,x_0)$ satisfies the conditions (A_1), (A_2) and (A_3).

Since $\psi|\dot{I}^{n+1}$ does not cover the point $(0,0,\dots,0,1)$ of S^{n-1}, the map $\psi|\dot{I}^{n+1}$ is inessential in S^{n-1}. Hence the map $\psi(I^{n+1},\dot{I}^{n+1})\to(S^{n+1},S^{n-1})$ is extendable to $\dot{I}^{n+1}$ such that $\psi(J^{n+1})\subset S^{n-1}$. Obviously the Brouwer's degree of the resultant map $\psi:\dot{I}^{n+2}\to S^{n+1}$ is ±1.

The composite map $H\circ\psi:\dot{I}^{n+2}\to X$ carries the subset J^{n+1} into the reference point x_0, and hence $H\circ\psi$ represents the same element of $\pi_{n+1}(X)$ with $H\circ\psi|I^{n+1}$. Consequently the map H satisfying the conditions (B_1), (B_2) and

(B_3) represents $\pm T(\alpha)=T(\alpha)\in \pi_{n+1}(X)/2\pi_{n+1}(X)$.

Let $h_0:(E^n, S^{n-1})\to(X, x_0)$ be a representative of $\alpha\in {}_2[\pi_n(X)]$, then there is a homotopy $h_t:(E^n, S^{n-1})\to(X, x_0)$ such that $h_1(x_1,\dots,x_n)=h_0(x_1,\dots,x_{n-1},-x_n)$, since h_1 represents $-\alpha$ and $\alpha=-\alpha$.

Define a map $H_0:(S^{n+1}, S^{n-1})\to(X, x_0)$ by setting

$$\begin{aligned} H_0(\phi(x_1,\dots,x_n,\theta)) &= h_{\frac{2\theta}{\pi}+\frac{1}{2}}(x_1,\dots,x_n) && -\pi/4\leqq\theta\leqq\pi/4\,,\\ &= h_{\frac{3}{2}-\frac{2\theta}{\pi}}(\rho_{(2\theta-\frac{\pi}{2})}(x_1,\dots,x_n)) && \pi/4\leqq\theta\leqq3\pi/4\,,\\ &= h_{\frac{3}{2}-\frac{2\theta}{\pi}}(x_1,\dots,-x_{n-1},x_n) && 3\pi/4\leqq\theta\leqq5\pi/4\,,\\ &= h_{\frac{2\theta}{\pi}-\frac{5}{2}}(\rho_{(\frac{3}{2}\pi-2\theta)}(x_1,\dots,x_{n-1},-x_n)) && 5\pi/4\leqq\theta\leqq7\pi/4\,, \end{aligned}$$

then H_0 satisfies the conditions (B_1), (B_2) and (B_3), and represents $T(\alpha)$.

Give a homotopy $H_t: S^{n+1}\to X$ by setting for $0\leqq t\leqq1$

$$\begin{aligned} H_t(\phi(x_1,\dots,x_n,\theta)) &= h_{(\frac{2\theta}{\pi}+\frac{1}{2})(1-t)}(x_1,\dots,x_n) && -\pi/4\leqq\theta\leqq\pi/4\,,\\ &= h_{(\frac{3}{2}+\frac{2\theta}{\pi})(1-t)}(\rho_{(2\theta-\frac{\pi}{2})}(x_1,\dots,x_n)) && \pi/4\leqq\theta\leqq3\pi/4\,,\\ &= h_{t+(\frac{5}{2}-\frac{2\theta}{\pi})(1-t)}(x_1,\dots,-x_{n-1},x_n) && 3\pi/4\leqq\theta\leqq5\pi/4\,,\\ &= h_{t+(\frac{2\theta}{\pi}-\frac{5}{2})(1-t)}(\rho_{(\frac{3}{2}\pi-2\theta)}(x_1,\dots,x_{n-1},-x_n)) && 5\pi/4\leqq\theta\leqq7\pi/4\,, \end{aligned}$$

and by setting for $1\leqq t\leqq2$

$$\begin{aligned} H_t(\phi(x_1,\dots,x_n,\theta)) &= h_0(\rho_{(\theta(t-1))}(x_1,\dots,x_n)) && -\pi/4\leqq\theta\leqq\pi/4\,,\\ &= h_0(\rho_{(\theta(t-1)+(2\theta-\frac{\pi}{2})(2-t)}(x_1,\dots,x_n)) && \pi/4\leqq\theta\leqq3\pi/4\,,\\ &= h_0(\rho_{(\theta(t-1)+\pi(2-t))}(x_1,\dots,x_n)) && 3\pi/4\leqq\theta\leqq5\pi/4\,,\\ &= h_0(\rho_{(\theta(t-1)+(2\theta+\frac{\pi}{2})(2-t))}(x_1,\dots,x_n)) && 5\pi/4\leqq\theta\leqq7\pi/4\,, \end{aligned}$$

then H_0 is homotopic to H_2 which is given by

$$H_2(\phi(x_1,\dots,x_n,\theta)) = h_0(\rho_{(\theta)}(x_1,\dots,x_n))\,.$$

Let $\omega: E^n\to S^n$ be a map given by

$$\begin{aligned} \omega(x_1,\dots,x_n) &= (2x_1,\dots,2x_n,\mu_-) && \text{for } \textstyle\sum x_i^2\leqq1/4\,,\\ &= (1-2x_1,\dots,1-2x_n,\mu_+) && \text{for } 1/4\leqq\textstyle\sum x_i^2\leqq1\,, \end{aligned}$$

where $\mu_-=-(1-4\sum x_i^2)^{\frac{1}{2}}$ and $\mu_+=(1-\sum(1-2x_i)^2)^{\frac{1}{2}}$. Then $\omega|E^n-S^{n-1}$ is a homeomorphism, and there is a map $h': S^n\to X$ such that $h'\circ\omega=h_0$, and h' represents $\pm\alpha$. Let $\bar{\mu}_n$ be the map given by

$$\bar{\mu}_n(\phi(x_1,\dots,x_n,\theta)) = \omega(r_\theta(x_1,\dots,x_n))\,,$$

then $H_2=h'\circ\bar{\mu}_n$.

For $n=2$, $\bar{\mu}_2: S^3\to S^2$ is the Hopf construction of a mapping $\bar{\mu}_2|\phi(S^1{}_{\frac{1}{2}}\times S^1)$ of type (ι_1,ι_1), where $S^1{}_{\frac{1}{2}}=\{(x_1,x_2)|x_1^2+x_2^2=1/2\}$, and $\phi|S^1{}_{\frac{1}{2}}\times S^1$ is a homeomorphism. Therefore the Hopf invariant of $\bar{\mu}_2$ is $\pm\iota_4$ and $\bar{\mu}_2$ represents $\pm\eta_2\in\pi_3(S^2)$.

For $n>2$, $\bar{\mu}_n: S^{n+1}\to S^n$ maps hemispheres E^{n+1}_{+1} and E^{n+1}_{-1} into hemispheres E^n_{+1} and E^n_{-1} respectively, and we have $\bar{\mu}_n(x_2,\dots,x_{n+1})=\bar{\mu}_{n-1}(x_2,\dots,x_{n+1}))$. Hence $\bar{\mu}_n$ is homotopic to $(-1)^n E(\bar{\mu}_{n-1})$, and by induction we see that $\bar{\mu}_n$ represents $\eta_n\in\pi_{n+1}(S^n)$.

Consequently we have

$$T(\alpha)=\{H_0\}=\{H_2\}=\{h'\circ\bar{\mu}_n\}=\pm\{\alpha\circ\eta_n\}=\{\alpha\circ\eta_n\} \text{ in } \pi_{n+1}(X)2/\pi_{n+1}(X)$$

and the proof of the theorem (5.4) is accomplished.

ii) Assume that elements $\alpha\in\pi_r(S^s)$, $\beta\in\pi_m(S^r)$ and $\gamma\in\pi_n(S^m)$ satisfy conditions $\alpha\circ\beta=0$, and $\beta\circ\gamma=0$. Let $f: S^r\to S^s$, $g: S^m\to S^r$ and $h: S^n\to S^m$ be representatives of α,β and γ respectively, and let $F_t: S^m\to S^s$ and $G_t: S^n\to S^r$ be nullhomotopies of $f\circ g=F_0$ and $g\circ h=G_0$. Define a map $H: S^{n+1}\to S^s$ by the rule

$$\text{(5.5)}\qquad \begin{aligned} H(d_n(x,t)) &= f(G_t(x)) && 0\leqq t\leqq 1,\\ &= F_{-t}(h(x)) && -1\leqq t\leqq 0. \end{aligned}$$

The construction of H depends on the choice of f, g, h, F_t and G_t. Let H' be another construction as above with respect to f', g', h', F'_t and G_t', and let f_t, g_t and h_t be homotopies from $f=f_0$, $g=g_0$ and $h=h_0$ to $f'=f_1$, $g'=g_1$ and $h'=h_1$ respectively. Define a homotopy $H_\tau: S^{n+1}\to S^s$ by

$$\begin{aligned} H_\tau(d_n(x,t)) &= f_\tau(G_{(2t-\tau)/(2-\tau)}(x)) && 0\leqq\tau/2\leqq t\leqq 1,\\ &= f_\tau(g_{\tau+2t}(h_{\tau+2t}(x))) && 0\leqq t\leqq\tau/2\leqq 1,\\ &= f_{\tau+2t}(g_{\tau+2t}(h_\tau(x))) && -1\leqq-\tau/2\leqq t\leqq 0,\\ &= F_{(\tau+2t)/(\tau-2)}(h_\tau(x)) && -1\leqq t\leqq-\tau/2\leqq 0, \end{aligned}$$

then $H_0=H$ and $H_1|S^n=H'|S^n$. Define two maps H_+ and $H_-: S^{n+1}\to S^s$ by

$$\begin{aligned} H_+(d_n(x,t)) &= H_1(d_n(x,t)) && 0\leqq t\leqq 1,\\ &= H'(d_n(x,-t)) && -1\leqq t\leqq 0, \end{aligned}$$

and

$$\begin{aligned} H_-(d_n(x,t)) &= H'(d_n(x,-t)) && 0\leqq t\leqq 1,\\ &= H_1(d_n(x,t)) && -1\leqq t\leqq 0, \end{aligned}$$

then H_+ represents an element of $\alpha\circ\pi_{n+1}(S^r)$ and H_- represents an element of $\pi_{m+1}(S^s)\circ E(\gamma)$, where $\alpha\circ\pi_{n+1}(S^s)$ and $\pi_{m+1}(S^r)\circ E(\beta)$ are subgroups of $\pi_{n+1}(S^s)$ consisted of the elements of the forms $\alpha\circ\zeta$ and $\xi\circ E(\beta)$ $(\zeta\in\pi_{n+1}(S^s), \xi\in\pi_{n+1}(S^s))$ respectively. As is easily seen, the sum $H_+ +(H'+H_-)$ is homotopic to H_1, and therefore

(5.6) *the class of H in $\pi_{n+1}(S^s)/\alpha\circ\pi_{n+1}(S^r)+\pi_{m+1}(S^s)\circ E(\gamma)$ depends only on α,β and γ, and it is denoted by $\{\alpha,\beta,\gamma\}$.*

Theorem (5.7) *If $\alpha\in{}_2[\pi_n(S^r)]$ and if $2<n<2r-2$, then*

$$\{2_r,\alpha,2_n\}=T(\alpha)=\{\alpha\circ\eta_n\} \quad \textit{in} \quad \pi_{n+1}(S^r)/2\pi_{n+1}(S^r).$$

Since the suspension homomorphism $E:\pi_{n-1}(S^{r-1})\to\pi_n(S^r)$ is an isomorphism onto for $n<2r-2$, there is an element α' of $\pi_{n-1}(S^{r-1})$ such that $E(\alpha')=\alpha$ and $2\alpha'=0$. By (2.5) we have $2\iota_r\circ\alpha=(\iota_r+\iota_r)\circ E(\alpha')=E(\alpha')+E(\alpha')=2\alpha=0$. Let $f':(S^{n-1},y_*)\to(S^{r-1},y_*)$ be a representative of α', and let $g_t':(S^{n-1},y_*)\to(S^{r-1},y_*)$ be a nullhomotopy of $f'\circ 2_{n-1}=g_0'$, where $2_m: S^m\to S^m (m>1)$ is a map of degree 2 given by

$$\begin{aligned}2_m(d_{m-1}(x,t)) &= d_{m-1}(x,2t-1) \qquad 0\leqq t\leqq 1\,,\\ &= d_{m-1}(x,2t+1) \quad -1\leqq t\leqq 0\,.\end{aligned}$$

Set $f=Ef'$ and $g_t=Eg_t'$, then f represents α and g_t is a nullhomotopy of $f\circ E2_{n-1}=g_0$. Let $f_t:(S^n,y_*)\to(S^r,y_*)$ be a nullhomotopy of $2_r\circ f=f_0$. Then $\{2\iota_r,\alpha,2\iota_n\}$ is represented by a map $H:(S^{n+1},y_*)\to(S^r,y_*)$ given by

$$\begin{aligned}H(d_n(x,t)) &= 2_r(g_t(x)) \qquad\quad 0\leqq t\leqq 1\,,\\ &= f_{-t}(E2_{n-1}(x)) \quad -1\leqq t\leqq 0\,.\end{aligned}$$

Now we shall calculate the composite map $H\circ\psi_{n+1}:(I^{n+1},\dot I^{n+1})\to(S^r,y_*)$ which is a representative of $\{2\iota_r,\alpha,2\iota_n\}$. Since $\psi_{m+1}(x_1,\dots,x_{m+1})=d_m(\psi_m(x_1,\dots,x_m),2x_{m+1}-1)$ and $2_m(d_{m-1}(x,t))=2_m(d_{m-1}(x,t+1))$ for $-1\leqq t\leqq 0$, we have

$$\begin{aligned}&H(\psi_{n+1}(x_1,\dots,x_{n+1})) = H(d_n(\psi_n(x_1,\dots,x_n),2x_{n+1}-1))\\ &=2_r(Eg'_{2t-1}(d_{n-1}(\psi_{n-1}(x_1,\dots,x_{n-1}),2x_n-1)))=2_r(d_{r-1}(g'_{2t-1}(\psi_{n-1}(x_1,\dots,x_{n-1}),2x_n-1))\\ &=2_r(d_{r-1}(g'_{2t-1}(\psi_{n-1}(x_1,\dots,x_{n-1})),2x_n+1))\\ &=H(\psi_{n+1}(x_1,\dots,x_{n-1},x_n+1/2,x_{n+1})) \qquad \text{for } 0\leqq x_n\leqq 1/2 \text{ and } 1/2\leqq x_{n+1}=t\leqq 1\,,\end{aligned}$$

$$\begin{aligned}&H(\psi_{n+1}(x_1,\dots,x_{n+1}))=H(d_n(\psi_n(x_1,\dots,x_n),2x_{n+1}-1))\\ &=f_{1-2t}(E2_{n-1}(d_{n-1}(\psi_{n-1}(x_1,\dots,x_{n-1}),2x_n-1)))\\ &=f_{1-2t}(d_{n-1}(2_{n-1}(d_{n-2}(\psi_{n-2}(x_1,\dots,x_{n-2}),2x_{n-1}-1)),2x_n-1))\\ &=f_{1-2t}(d_{n-1}(2_{n-1}(d_{n-2}(\psi_{n-2}(x_1,\dots,x_{n-2}),2x_{n-1}+1)),2x_n-1))\\ &=H(\psi_{n+1}(x_1,\dots,x_{n-1}+1/2,x_n,x_{n+1})) \qquad \text{for } 0\leqq x_{n-1}\leqq 1/2 \text{ and } 0\leqq x_{n-1}=t\leqq 1/2.\end{aligned}$$

and $\quad H(\psi_{n+1}(x_1,\dots,x_{n-2},x_{n-1}/2,x_n/2,1/2))$

$$\begin{aligned}&= H(d_n(\psi_n(x_1,\dots,x_{n-2},x_{n-1}/2,x_n/2),0)\\ &= (2_r\circ Ef'\circ E2_{n-1})(d_{n-1}(\psi_{n-1}(x_1,\dots,x_{n-2},x_{n-1}/2),x_n-1))\\ &= 2(d_{r-1}(f'(2_{n-1}(d_{n-2}(\psi_{n-2}(x_1,\dots,x_{n-2}),x_{n-1}-1)),x_n-1))\\ &= d_{r-1}(f'(d_{n-2}(\psi_{n-2}(x_1,\dots,x_{n-2}),2x_{n-1}-1))2x_n-1)\\ &= d_{r-1}(f'(\psi_{n-1}(x_1,\dots,x_{n-1})),2x_n-1)\\ &= f(\psi_n(x_1,\dots,x_n)).\end{aligned}$$

Therefore H satisfies the conditions (A_1), (A_2) and (A_3), and represents $T(\alpha)$. Consequently we obtain $\{2\iota_r,\alpha,2\iota_n\}=\{H\}=T(\alpha)$ in $\pi_{n+1}(S^r)/2\pi_{n+1}(S^r)$.

Lemma (5.8) *If elements $\alpha\in\pi_r(S^s)$, $\beta\in\pi_m(S^r)$, $\gamma\in\pi_n(S^r)$ and $\delta\in\pi_l(S^n)$ satisfy $\alpha\circ\beta=0$, $\beta\circ\gamma=0$, and $\gamma\circ\delta=0$, then*

$$\alpha\circ\{\beta,\gamma,\delta\}=\{\alpha,\beta,\gamma\}\circ E(-\delta) \quad \textit{in} \quad \pi_{l+1}(S^s)/\alpha\circ\pi_{n+1}(S^r)\circ E(\delta).$$

In the lemma, $\alpha \circ \{\beta, \gamma, \delta\}$ and $\{\alpha, \beta, \gamma\} \circ E(-\delta)$ are classes of $\alpha \circ \zeta$ and $\xi \circ E(-\delta)$ (for elements $\zeta \in \{\beta, \gamma, \delta\}$ and $\xi \in \{\alpha, \beta, \gamma\}$) respectively in the factor group $\pi_{l+1}(S^s)/\alpha \circ (\beta \circ \pi_{l+1}(S^m) + \pi_{n+1}(S^r) \circ E(\delta)) = \pi_{l+1}(S^s)/\alpha \circ \pi_{n+1}(S^r) \circ E(\delta)$ $= \pi_{l+1}(S^s)/(\alpha \circ \pi_{n+1}(S^r) + \pi_{m+1}(S^s) \circ E(\gamma)) \circ E(-\delta)$. Let f, g, h and k be representatives of α, β, γ and δ, and let F_t, G_t and H_t be nullhomotopies of $f \circ g, g \circ h$ and $h \circ k$ respectively. Consider a homotopy $K_\tau : S^{l+1} \to S^s$ which is given by

$$\begin{aligned} K_\tau(d_l(x, t)) &= f(G_t(k(x))) & 0 \leqq t \leqq 1\,, \\ &= F_{t(\tau-1)}(H_{-t\tau}(x)) & -1 \leqq t \leqq 0\,, \end{aligned}$$

then K_0 represents $\{\alpha, \beta, \gamma\} \circ E(-\delta)$ and K_1 represents $\alpha \circ \{\beta, \gamma, \delta\}$, and it follows from this that we have the lemma.

iii) In this section we shall use the notations of the previous section and assume that $\alpha \circ \beta = 0$, and $\beta \circ \gamma = 0$.

Let $K_\alpha^{r+1} = S^s \cup e^{r+1}$ be a cell complex, in which e^{r+1} is attached to S^s by a characteristic map $\tilde{\alpha} : (E_+^{r+1}, S^r) \to (K_\alpha^{r+1}, S^s)$ such that $\tilde{\alpha} | S^r = f$ represents α. Define a mapping, $\tilde{g} : S^{m+1} \to K_\alpha^{r+1}$ by setting

$$\begin{aligned} \tilde{g}(d_m(x, t)) &= \tilde{\alpha}(d_m(g(x), t)) & 0 \leqq t \leqq 1\,, \\ &= F_{-t}(x) & -1 \leqq t \leqq 0\,, \end{aligned}$$

then $\tilde{g}$ represents an element $\tilde{\beta}$ of $\pi_{m+1}(K_\alpha^{r+1})$.

Lemma (5.9) *$\tilde{g} \circ E(h)$ is homotopic to a mapping $S^{n+1} \to S^s$ which represents $\{\alpha, \beta, \gamma\}$.*

The lemma follows from a homotopy H_τ given by

$$\begin{aligned} H_\tau(d_n(x, t)) &= \tilde{\alpha}(d_n(G_{t\tau}(x), t)) & 0 \leqq t \leqq 1\,, \\ &= F_{-t}(h(x)) & -1 \leqq t \leqq 0\,. \end{aligned}$$

iv) For example, consider an element $\zeta \in \pi_{r+3}(S^r)$ of $\{\eta_r, 2\iota_{r+1}, \eta_{r+1}\}$, then from (5.8) we have $2\zeta = \zeta \circ 2\iota_{r+3} = \eta_r \circ \xi$ for an element ξ of $\{2\iota_{r+1}, \eta_{r+1}, 2\iota_{r+2}\}$, and from (5.7) $\{2\iota_{r+1}, \eta_{r+1}, 2\iota_{r+2}\} = \eta_{r+1} \circ \eta_{r+2}$ in $\pi_{r+3}(S^{r+1})$ $(r \geqq 3)$. Therefore

Lemma (5.10) *There is an element ζ of $\pi_{r+3}(S^r)$ such that $2\zeta = \eta_r \circ \eta_{r+1} \circ \eta_{r+2} \neq 0$, and ζ has order 4. $(r \geqq 3)$.*

Chapter 6. Eilenberg-MacLane complex.

Let $K(\Pi, n)$ be the complex of a (abelian) group Π which is defined and treated by S. Eilenberg and S. MacLane [7]. A q-cell of $K(\Pi, n)$ is an n-dimensional cocycle $\sigma^q \in Z_n(\Delta_q ; \Pi)$ of the q-dimensional ordered simplex Δ_q. The ***suspension homomorphism*** $S : H_q(K(\Pi, n)) \to H_{q+1}(K(\Pi, n+1))$ is given by setting $S\sigma^q = T\sigma^q - \sigma_0^{q+1}$, where $\sigma_0^{q+1}(\Delta) = 0 \in \Pi$ and $T\sigma^q$ is defined for each $(n+1)$-dimensional ordered subsimplex $(r_0, \dots, r_{n+1})$ of $\Delta_{q+1} = (0, \dots, q+1)$ such as

$$\begin{aligned} T\sigma^q(r_0, \dots, r_{n+1}) &= \sigma^q(r_0, \dots, r_n) & \text{if } r_{n+1} = q+1\,, \\ &= 0 & \text{if } r_{n+1} < q+1\,. \end{aligned}$$

S. Eilenberg and S. MacLane reduced the complex $K(\Pi, n)$ to $A(\Pi, n)$ and calculated the following results for the infinite cycle group Z,

$$
(6.1)\quad
\begin{array}{llll}
H_{n+1}(K(Z,n)) = 0 & n \geqq 1, & H_{n+2}(K(Z,n)) = Z_2 & n \geqq 3, \\
H_{n+3}(K(Z,n)) = 0 & n \geqq 4, & H_{n+4}(K(Z,n)) = Z_2 + Z_3 & n \geqq 5, \\
H_{n+5}(K(Z,n)) = 0 & n \geqq 6, & H_{n+6}(K(Z,n)) = Z_2 + Z_2 & n \geqq 7, \\
H_{n+7}(K(Z,n)) = 0 & n \geqq 8, & H_{n+8}(K(Z,n)) = Z_2 + Z_2 + Z_3 + Z_5 & n \geqq 9, \\
H_{n+9}(K(Z,n)) = Z_2 & n \geqq 10. & &
\end{array}
$$

In particular, $H_{n+4}(K(Z,n))$ are calculated for lower dimensions:

(6.2) *$H_6(K(Z,2))=Z$, $H_7(K(Z,3))=Z_3$, $H_8(K(Z,4))=Z+Z_3$ and the suspension homomorphism $S: H_6(K(Z,2)) \to H_7(K(Z,3))$ is onto and the $(n-3)$-fold suspension $S^{n-3}: H_7(K(Z,3)) \to H_{n+4}(K(Z,n))$ is an isomorphism into.*

Let K_n be a CW-complex such that its $(n+1)$-skeleton is an n-sphere S^n and homotopy groups $\pi_i(K_n)$ for $i>n$ vanish. The existence of such a complex was shown by J. H. C. Whitehead [24]. Furthermore we may assume that the $(n+k)$-skeleton K_n^{n+k} of K_n is a finite cell complex, for the homotopy groups of a finite complex are finitely generated (*cf.* [14]). Therefore the singular homology groups of K_n coincide to the usual homology groups.

Consider the suspended space $'K=E(K_n)$ of K_n with reference point $y_* \in S^n \subset K_n$, then $'K$ is also a cell complex and its $(n+2)$-skeleton is S^{n+1}.

Let $\chi: (E^{n+k+1}, S^{n+k}) \to ('K^{n+k} \cup e^{n+k+1}, 'K^{n+k})$ be a characteristic map of a cell $e^{n+k+1} \in 'K$, and let a mapping $f: 'K^{n+k} \to K_{n+1}^{n+k}$ $(k \geqq 2)$ be given, then the composite map $f \circ (\chi | S^{n+k})$ represents an element of $\pi_{n+k}(K_{n+k}^{n+k})$. Since $\pi_{n+k}(K_{n+1}^{n+k+1}) = \pi_{n+k}(K_{n+1}) = 0$ for $k \geqq 2$, there is a mapping $\chi': (E^{n+k+1}, S^{n+k}) \to (K_{n+1}^{n+k+1}, K_{n+1}^{n+k})$ such that $\chi' | S^{n+k} = f \circ (\chi | S^{n+k})$. A mapping $\chi' \circ \chi^{-1}$ defines an exte sion of f over e^{n+k+1}. By induction we obtain a mapping

$$f_0 : E(K_n) \to K_{n+1}$$

such that $f_0 | S^n$ is the identical map and f_0 maps the $(n+k)$-skeleton of $E(K_n)$ into the $(n+k)$-skeleton of K_{n+1}.

Let $S_{n-1}(K_n)$ be a subcomplex of the singular complex $S(K_n)$ of K_n consisted of the simplexes $T^q: \Delta_q \to K_n$ such that T maps the $(n-1)$-subsimplex of Δ_q into y_*. Define a *suspended* simplex $E'(T^q): \Delta_{q+1} \to E(K_n)$ of T^q by setting $E'(T^q)(\lambda_0, \dots, \lambda_{q+1}) = d(T(\lambda_0, \dots, \lambda_q), 2\lambda_{q+1} - 1)$ for the barycentric representative $(\lambda_0, \dots, \lambda_{q+1})$ of a point of Δ_{q+1}. Define a chain transformation $S': S_{n-1}(K_n) \to S_n(K_{n+1})$ by setting $S'(T^q) = f_0^*(E'(T^q)) - T_0^{q+1}$ where $T_0^{q+1}(\Delta_{q+1}) = y_*$, then we obtain a suspension homomorphism $S': H^q(S_{n-1}(K_n)) \to H_{q+1}(S_n(K_{n+1}))$.

Lemma (6.3) *Let $\kappa_n: K(Z, n) \to S_{n-1}(K_n) \subset S(K_n)$ be the natural chain equivalence given in* [6], *then we can choose a natural chain equivalence*

$\kappa_{n+1}: K(Z, n+1) \to S_n(K_{n+1})$ *such that* $S' \circ \kappa_n = \kappa_{n+1} \circ S$.

Therefore we have a commutative diagram

$$\begin{tikzcd}
H_q(K(Z,n)) \arrow[rr, "S"] \arrow[d, "\kappa_n^*"] & & H_{q+1}(K(Z,n+1)) \arrow[d, "\kappa_{n+1}^*"] \\
H_q(K_n) \arrow[rr, "S'"] \arrow[dr, "E"'] & & H_{q+1}(K_{n+1}) \\
& H_{q+1}(E(K_n)) \arrow[ur, "f_0"'] &
\end{tikzcd}$$

where κ_n^*, κ_{n+1}^* and E are isomorphisms.

Now we shall prove an important lemma:

Lemma (6.4) $H_{n+k+1}(K(Z, n)) \approx \pi_{n+k}(K_n^{n+k-1})/\partial[\pi_{n+k+1}(K_n^{n+k}, K_n^{n+k-1})]$ $(k \geqq 1)$.

In the following diagram

$$\begin{tikzcd}
& & \pi_{n+k}(K_n^{n+k}, K_n^{n+k-1}) & & \\
\pi_{n+k+2}(K_n^{n+k+2}, K_n^{n+k+1}) \arrow[r, "\partial_2"] & \pi_{n+k+1}(K_n^{n+k+1}, K_n^{n+k}) \arrow[ur, "\partial_3"] \arrow[r, "i_2^*"'] & \pi_{n+k+1}(K_n^{n+k+2}, K_n^{n+k}) \arrow[u, "\partial_1"] \arrow[r, "j_2"] & \pi_{n+k+1}(K_n^{n+k+2}, K_n^{n+k+1}) & \\
& \pi_{n+k+1}(K_n^{n+k+2}) \arrow[r] & \pi_{n+k+1}(K_n^{n+k+2}, K_n^{n+k-1}) \arrow[u, "j_1"] \arrow[ur, "j^3"] \arrow[r, "\partial_0"] & \pi_{n+k}(K_n^{n+k-1}) \arrow[r] & \pi_{n+k}(K_n^{n+k+2}) \\
& & \pi_{n+k+1}(K_n^{n+k}, K_n^{n+k-1}) \arrow[u, "i_1^*"] \arrow[ur, "\partial"] & &
\end{tikzcd}$$

the exactness of each direct sequences and the commutativity relations hold. By a simple algebraic lemma of T. Kudo [13, II, lemma 1], there is an isomorphism: *kernel* ∂_3 / *kernel* $i_2^* \approx$ *kernel* j_3 / *kernel* j_1. Since $\pi_{n+k+1}(K_n^{n+k+2}, K_n^{n+k+1}) = 0$, $\pi_{n+k+1}(K_n^{n+k+2}) = \pi_{n+k+1}(K_n) = 0$ and $\pi_{n+k}(K_n^{n+k+1}) = \pi_{n+k}(K_n) = 0$, we have $H_{n+k+1}(K_n) =$ *kernel* ∂_3/*image* $\partial_2 =$ *kernel* ∂_3/*kernel* $i_2^* \approx$ *kernel* j_3/*kernel* $j_1 = \pi_{n+k+1}(K_n^{n+k}, K_n^{n+k-1})$/*image* $i_1^* \approx \pi_{n+k}(K_n^{n+k-1})$/*image* ∂, and the proof ot lemma is established.

Remark that in the lemma (6.4), K_n^{n+k} is the $(n+k)$-skeleton of K_n, but we may assume that K_n^{n+k} is an $(n+k)$-demensional complex such that its $(n+1)$-skeleton is S^n and $\pi_i(K^{n+k}) = 0$ for $n < i < n+k$, because we can construct a complex K_n whose $(n+k)$-skeleton is K_n^{n+k}.

For $n > 3$, we define a cell complex K_n^{n+3} as follows.

$(6.5)_1$ $K_n^{n+3} = S^n \cup e^{n+2} \cup e^{n+3}$.

$(6.5)_2$ e^{n+2} is attached to S^n by a characteristic map $\tilde{\eta}_n: (E^{n+2}, S^{n+1}) \to (S^n \cup e^{n+2}, S^n)$ such that $\tilde{\eta}_n | S^{n+1}$ represents $\eta_n \in \pi_{n+1}(S^n)$ and $E(\tilde{\eta}_n | S^{n+1}) = \tilde{\eta}_{n+1} | S^{n+2}$, then we have $E(K_n^{n+2}) = K_{n+1}^{n+3}$ for $n \geqq 2$.

$(6.5)_3$ e^{n+3} is attached to $S^n \cup e^{n+2} = K_n^{n+2}$ by a characteristic map $\tilde{\zeta}_n: (E^{n+3}, S^{n+2}) \to (K_n^{n+2} \cup e^{n+3}, K_n^{n+3})$ where $\tilde{\zeta}_n | S^{n+2}$ is given as follows. For $n=3$, let $\bar{2}: (E_+^5, S^4) \to (E^5, S^4)$ be a mapping of degree 2, then there is a mapping $h: E_-^5 \to S^3$ such that $h | S^4 = (\tilde{\eta}_3 \circ \bar{2}) | S^4$ for $2\eta_n = 0$. The mapping $\tilde{\zeta}_3 | S^5$ is defined by setting $\tilde{\zeta}_3 | E_-^5 = \tilde{\eta}_3 \circ \bar{2}$ and $\tilde{\zeta}_3 | E_-^5 = h$. For $n > 3$, $\tilde{\zeta}_n | S^{n+2}$ is defined by setting $\tilde{\zeta}_n | S^{n+2}$

$=E(\tilde{\zeta}_{n-1}|S^{n+1})$ inductively, then $E(K_n^{n+3})=K_{n+1}^{n+4}$. It is easily verified that a generator $\tilde{\zeta}_n$ of $\pi_{n+2}(K_n^{n+2})$ is represented by $\zeta_n|S^{n+2}$ for $n\geqq 3$, and that

(6.6) $\qquad \pi_{n+1}(K_n^{n+2})=0$ *and* $\pi_{n+2}(K_n^{n+3})=0 \quad$ for $n\geqq 3$.

By (5.9), $(\tilde{\zeta}_n|S^{n+2})\circ\eta_{n+2}$ is homotopic to an representative ζ of $\{\eta_n, 2\iota_{n+1}, \eta_{n+1}\}$, in K_n^{n+2} and by (5.10) we have $2\zeta=\eta_n\circ\eta_{n+1}\circ\eta_{+2}$. Since a generator of the image of $\partial:\pi_{n+4}(K_n^{n+2}, S^n)\to\pi_{n+3}(S^n)$ is $\eta_n\circ\eta_{n+1}\circ\eta_{n+2}$, and a generator of the image of $\partial:\pi_{n+4}(K_n^{n+3}, K_n^{n+2})\to\pi_{n+3}(K_n^{n+2})$ is ζ, and since they are not trival, we obtain

(6.7) *The boundary homomorphisms* $\partial:\pi_{n+4}(K_n^{n+3}, K_n^{n+2})\to\pi_{n+3}(K_n^{n+2})$ *and* $\partial:\pi_{n+4}(K_n^{n+2}, S^n)\to\pi_{n+3}(S^n)$ *are isomorphisms into for* $n\geqq 3$.

Chapter 7. The group $\pi_{n+3}(S^n)$.

Applying the lemma (6.4) to the complex K_n^{n+3} of (6.5), we have from (6.1) $\; Z_2+Z_3\approx\pi_{n+4}(K_n^{n+2})/\partial(\pi_{n+4}(K_n^{n+3}, K_n^{n+2}))\;$ for $n\geqq 5$.
By (6.7), $\partial:\pi_{n+4}(K_n^{n+3}, K_n^{n+2})\to\pi_{n+3}(K_n^{n+2})$ is isomorphic and $\pi_{n+3}(K_n^{n+2})$ must have 12 elements. In the exact sequence $\pi_{n+4}(K_n^{n+2}, S^n)\xrightarrow{\partial}\pi_{n+3}(S^n)\xrightarrow{i^*}\pi_{n+3}(K_n^{n+2})\xrightarrow{j}\pi_{n+3}(K_n^{n+2}, S^n)\xrightarrow{\partial'}\pi_{n+2}(S^n)$, ∂' is isomorphism onto and hence i^* is onto, while (6.7) shows that ∂ is isomorphism into and therefore $\pi_{n+3}(S^n)$ must have 24 elements. By (4.3) $\pi_6(S^3)$ has 12 elements and by (5.10) it contains an element of order four, therefore $\pi_6(S^3)$ is cyclic group of order 12. The only element of order 2 in $\pi_6(S^3)$ is $\eta_3\circ\eta_4\circ\eta_5$ and its Hopf invariant is trivial, then α_3 must have order 4 or 12 for $H(\alpha_3)=\eta_6\neq 0$. Since $\alpha_n=2\nu_n$ for $n\geqq 5$, ν_n has order 8 or 24, and we obtain.

Proposition (7.1) $\;\pi_6(S^3)=Z_{12}, \pi_7(S^4)=Z+Z_{12}$ *and* $\pi_{n+3}(S^n)=Z_{24}$ *for* $n\geqq 5$.

Now we shall calculate generators of these groups.

Let M^{2n} be the complex projective space, and let $M^{2n}=S^2\cup e^4\cup\cdots\cup e^{2n}$ be its cell decomposition as in (4.iv) with characteristic maps $\tilde{p}_{n-1}:(E^{2n}, S^{2n-1})\to(M^{2n}, M^{2n-2})$ such that $p_{n-1}=\tilde{p}_{n-1}|S^{2n-1}$ are fibre maps with fibre S^1. Then p_n induces isomorphisms $p_n^*:\pi_p(S^{2n+1}, S^1)\to\pi_p(M^{2n})$. Let K_2 be the limit space $\bigcup_n M^{2n}$, then $M^{2n}=K_2^{2n}$ and K_2^4 is the complex given in (6.5), and the homotopy groups of K_2 are trivial except $\pi_2(K_2)=Z$. Next we construct a complex K_3^7 whose i-th homotopy groups vanish for $3<i<7$. In the exact sequence: $\pi_7(K_3^5, S^3)\xrightarrow{\partial}\pi_6(S^3)\to\pi_6(K_3^5)\to\pi_6(K_3^5, S)\xrightarrow{\partial'}\pi_5(S^3)$, ∂ and ∂' are isomorphisms into and $\pi_6(S^3)\approx Z_{12}$, hence $\pi_6(K_3^5)\approx Z_6$ and its generator is represented by a mapping $g: S^6\to S^3$ which represents a generator of $\pi_6(S^3)$. Define $K_3^7=K_3^6\cup e^7$ with characteristic mapping $\tilde{g}:(E^7, S^6)\to(S^3\cup e^7, S^3)$ such that $\tilde{g}|S^3=g$, then $\pi_6(K_3)=0$ for $3<i<7$, and we can construct the complex K_3 such that its 7-skeleton is K_3^6.

Let $f_0: E(K_2)\to K_3$ be a mapping given in Chapter 6, then (6.3) and (6.2) shows that $f_0^*: H_7(E(K_2))=Z\to H_7(K_3)=Z_3$ is onto, and therefore f_0 maps $E(e^6)$ onto e^7 with degree k, where k is prime to 3. This implies that $E\tilde{p}_2: S^6\to K_3^5=E(M^4)$ represents an element of degree 3 or 6 in $\pi_6(K_3^5)$. Consider a diagram

$$\begin{array}{ccccccc}\pi_5(S^2) & \longrightarrow & \pi_5(K_2^4) & \xrightarrow{j} & \pi_5(K_2^4, S^2) & \longrightarrow & \pi_4(S^2)\\ & & \downarrow E & & \downarrow E' & & \\ \pi_6(S^3) & \longrightarrow & \pi_6(K_3^5) & \longrightarrow & \pi_6(K_3^5, S^3). & & \end{array}$$

From (3.14), $\pi_6(K_2^4, S^2)$ has direct factor isomorphic to Z, and its generator is the relative product $[\iota_2, \iota_4]_r$ for generators $\iota_2\in\pi_2(S^2)$ and $\iota_4\in\pi_4(K_2^4, S^2)$. Since $\pi_5(K_2^4)\approx\pi_5(S^5)=Z$, $\pi_4(S^2)=Z_2$ and $[\iota_2, \eta_2]=0$[10] there is an element γ of $\pi_5(K_2^4)$ such that $j(\gamma)=[\iota_2, \iota_4]$ and $E(\gamma)$ is an element of $\pi_6(K_3^5)$ which has order 3 or 6. By lemma (2.32) α_3 is an element of $\pi_6(S^3)$ such that $i^*(\alpha_3)=E(\gamma)$. Consequently α_3 must have order 12 and generate $\pi_6(S^3)$. By (4.3) we have that ν_n has order 24 and generates $\pi_{n+3}(S^n)$ for $n\geqq 5$. We have

Theorem (7.2) i) $\pi_6(S^3)\approx Z_{12}$ *and its generator is* α_3,

ii) $\pi_7(S^4)\approx Z+Z_{12}$ *and its generators are* ν_4 *and* α_4,

iii) $\pi_{n+3}(S^n)\approx Z_{24}$ *for* $n\geqq 5$ *and its generator is* ν_n.

And also we have relations

(7.3) $6\nu_n=3\alpha_n=\zeta_n\circ\eta_{n+2}$ *in* K_n^{n+2}, $6\nu_n\in\{\eta_n, 2\iota_{n+1}, \eta_{n+1}\}$ *and* $12\nu_n=6\alpha_n=\eta_n\circ\eta_{n+1}\circ\eta_{n+2}$ *for* $n\geqq 5$.

Chapter 8. Caluculations in higher dimensions.

i) In this chapter, our calculations are teated for sufficiently large values of n, such that the exision theorem (1.23) holds, for example we may assume $n>10$.

Define a cell complex $K_n^{n+6}=K_n^{n+3}\cup e^{n+4}\cup e^{n+5}\cup e_1^{n+6}\cup e_2^{n+6}$ as follows.

$(8.1)_1$ $K_n^{n+3}=S^n\cup e^{n+2}\cup e^{n+3}$ is defined as in (6.5).

$(8.1)_2$ e^{n+4} is attaced to $S^n\subset K_n^{n+3}$ by a characteristic map $\tilde{\nu}_n: (E^{n+4}, S^{n+3})\to(S^n\cup e^{n+4}, S^n)$ such that $\tilde{\nu}_n|S^{n+3}$ represents the generator ν_n of $\pi_{n+3}(S^n)$.

$(8.1)_3$ e^{n+5} is attached to K_n^{n+4} by a characteristic may $\tilde{\xi}_n: (E^{n+5}, S^{n+4})\to(K_n^{n+5}, K_n^{n+4})$ as follows, set $\tilde{\xi}_n|E_{++}^{n+4}=\tilde{\nu}_n\circ\tilde{6}$ where $E_{++}^{n+4}=d_{n+3}(S^{n+3}\times[1/2, 1])$ and $\tilde{6}: (E_{++}^{n+4}, \dot{E}_{++}^{n+4})\to(E^{n+4}, S^{n+3})$ is a mapping of degree 6, and set $\tilde{\xi}_n|E_-^{n+4}=\tilde{\zeta}_n\circ\bar{\eta}_{n+3}$ where $\bar{\eta}_{n+3}: (E_-^{n+4}, S^{n+3})\to(E^{n+3}, S^{n+2})$ repiesents generator $\partial^{-1}\eta_{n+2}$ of $\pi_{n+4}(E^{n+3}, S^{n+2})$, then we can extend the mapping $\tilde{\xi}_n$ over the subset E_+^{n+4}—Int. E_{++}^{n+4} into K_n^{n+2} for $6\nu_n=\zeta_n\circ\eta_{n+2}$ in K_n^{n+2}.

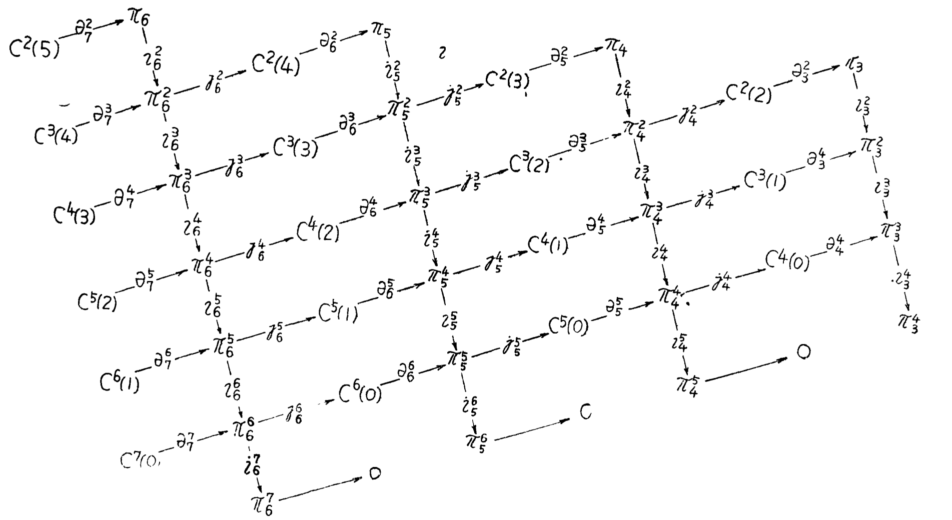
$(8.1)_4$ e_1^{n+6} is attaced to $S^n\cup e^{n+4}\subset K_n^{n+5}$ by a characteristic map $\tilde{\eta}_{n+4}: (E^{n+6}, S^{n+5})\to(K_n^{n+6}, K_n^{n+5})$ as follows, set $\tilde{\eta}_{n+4}|E_+^{n+5}=\tilde{\nu}_n\circ\bar{\eta}_{n+4}$ where $\bar{\eta}_{n+4}: (E_+^{n+5}, S^{n+4})\to(E^{n+4}, S^{n+3})$ represents generator $\partial^{-1}\eta_{n+3}$ of $\pi_{n+5}(E^{n+4}, S^{n+3})$, and extend the mapping $\tilde{\eta}_{n+1}|S^{n+4}: S^{n+4}\to S^n$ over E_-^{n+5} such that $\eta_{n+4}(E_-^{n+5})\subset S^n$ for $\nu_n\circ\eta_{n+3}=0$.

10) Since $g: S^3\times S^2\to S^2$ in iii) of Ch. 4 has type (η_2, ι_2), we have $[\iota_2, \eta_2]=-[\eta_2, \iota_2]=0$ by (2.26).

$(8.1)_5$ e_2^{n+6} is attached to $S^n \cup e^{n+2} \subset K_n^{n+5}$ by a characteristic may $\tilde{\nu}_{n+2}: (E^{n+6}, S^{n+5}) \to (K_n^{n+6}, K_n^{n+5})$ as follows, set $\tilde{\nu}_{n+2} | E_+^{n+5} = \eta_n \circ \bar{\nu}_{n+2}$ where $\bar{\nu}_{n+2}: (E_+^{n+5}, S^{n+4}) \to (E^{n+2}, S^{n+1})$ represents the generator $\partial^{-1}\nu_{n+1}$ of $\pi_{n+5}(E^{n+2}, S^{n+1})$, and extend the mapping $\tilde{\nu}_{n+2} | S^{n+4}: S^{n+4} \to S^n$ over E_-^{n+5} such that $\tilde{\nu}_{n+2}(E_-^{n+5}) \subset S^n$ for $\eta_n \circ \nu_{n+1} = 0$.

For convenience, we shall use the following notations in the chapter: $\pi_r^t = \pi_{n+r}(K_n^{n+t})$, $\pi_r = \pi_{n+r}(S^n)$ and $C^t(r-t) = \pi_{n+r}(K_n^{n+t}, K_n^{n+t-1})$. By (1.27), $C^t(r-t)$ is isomorphic to the $(n+t)$-dimensional chain group with coefficent group π_{r-t} for sufficiently large n.

In a diagram



subsequences $\cdots \to \pi_r^{s-1} \xrightarrow{i_r^s} \pi_r^s \xrightarrow{j_r^s} C^s(r-s) \xrightarrow{\partial_r^s} \cdots \to \pi_{s-1}^{s-1} \to \pi_{s-1}^s \to 0$ are exact, and the composite homomorphisms $C^s(r-s) \to \pi_{r-1}^{s-1} \to C^{s-1}(r-s)$ are the boundary homomorphism of the chain groups.

We already know that $\pi_3^1 = \pi_3 = Z_{24}$, $\pi_3^2 = Z_{12}$, $\pi_3^3 = Z_6$ and the injection homomorphisms: $\pi_3 \xrightarrow{i_3^2} \pi_3^2 \xrightarrow{i_3^3} \pi_3^3$ are onto,

ii) The image of ∂_4^4 is generated by $\partial\{\tilde{\nu}_n\} = \nu_n$, and ν_n is the generator of π_3, π_3^2 and π_3^3, hence ∂_4^4 is onto and $\pi_3^4 = 0$. The complex K_n^{n+4} has the homotopy groups $\pi_i(K_n^{n+4}) = 0$ for $n < i < n+4$ and we have from (6.4) and (6.1)

$$0 = H_{n+5}(K_n) \approx \pi_4^3 / image\ \partial_5^4 .$$

Hence ∂_5^4 is onto and i_4^4 is trivial. The image of ∂_5^4 is generated by $\nu_n \circ \eta_{n+3} = 0$,[11] therefore we have $\pi_4^3 = 0$. The image of ∂_5^3 is generated by $\zeta_n \circ \eta_{n+2} \circ \eta_{n+3}$[12] $= 6\nu_n \circ \eta_{n+3} = 0$ and hence $\pi_4^2 = 0$. The image of ∂_5^2 is generated by $\eta_n \circ \nu_{n+1} = 0$[11] and hence $\pi_4 = 0$. Consequently we have

11) Cf. (4.4) and (4.5).
12) Cf. (7.3).

Proposition (8.3) $\pi_{n+4}(S^n)=0$ *for* $n\geqq 6$.

iii) Since $\pi_4^3=0$, π_4^4 is isomorphic to the kernel of ∂_4^4. $\pi_3^3=Z_6$, $C^4(0)=Z$ and ∂_4^4 is onto, so we have that a generator of ∂_4^4 is represented by a mapping of degree 6. Let an element ξ_n of π_4^4 be presented by $\tilde{\xi}_n|S^{n+4}$, then $j_4^4(\xi_n)$ is represented by a mapping of degree 6 and therefore $j_4^4(\xi_n)$ generates the kernel of ∂_4^4. Consequently we have that ξ_n generates π_4^4 and ∂_5^5 is onto, hence $\pi_4^5=0$. Applying (6.4) and (6.1) to the complex K_n^{n+5} which has the homotopy groups $\pi_i(K_n^{n+5})=0$ for $n<i<n+5$, we have

$$Z_2+Z_2=H_{n+6}(K_n)\approx\pi_5^4/image\ \partial_6^5.$$

The image of ∂_6^5 is generated by $\xi_n\circ\eta_{n+5}$. Since incidence number $[e^{n+5}:e^{n+4}]=6$, we have $j_5^4(\xi_n\circ\eta_{n+5})=0$, hence there is an element ξ' of π_5^3 such that $i_5^4(\xi')=\xi_n\circ\eta_{n+5}$. From the structure of the mapping $\tilde{\xi}_n$, we have easily that the *imag* $j_5^3(\xi')$ is the non-zero element of $C^3=Z_2$, hence $\xi_n\circ\eta_{n+5}=0$, *image* $\partial_6^5=Z_2$ and the group π_5^4 must have form $Z_2+Z_2+Z_2$ or Z_2+Z_4. Let $\eta'\in\pi_5^4$ be represented by, $\tilde{\eta}_{n+4}|S^{n+5}$, then $j_5^4(\eta')$ is the generator of $C^4(1)=Z_2$. If $2\eta'\neq 0$, we have $2\eta'=\xi_n\circ\eta_{n+5}$. The mapping $\tilde{\eta}_{n+4}|S^{n+5}$, however, does not cover the cell e^{n+3}, therefore we have $2\eta'=0$ in K_n^{n+4} and $\pi_5^4=Z_2+Z_2+Z_2$. Since the image of ∂_6^4 is generated by $\nu_n\circ\eta_{n+3}\circ\eta_{n+4}=0$, we have $\pi_5^3=Z_2+Z_2$, j_5^3 is onto and image $i_5^3=Z_2$. Let ν_1 be a generator of $C^3(3)=Z_{24}$. Since the incidence number $[e^{n+3}:e^{n+2}]=2$, we can chose a generator ν_2 of $C^2=Z_{24}$ such that $j_5^2\circ\partial(\nu_1)=2\nu_2$, hence *image* $\partial_6^3=Z_{24}$ or Z_{12}. Since $12(\zeta_n\circ\nu_{n+2})=\zeta_n\circ 12\nu_{n+2}=\zeta_n\circ\eta_{n+2}\circ\eta_{n+3}\circ\eta_{n+4}$[12] $=6\nu_n\circ\eta_{n+3}\circ\eta_{n+4}=0$ in K_n^{n+1}, we have *image* $\partial_6^3=Z_{12}$ and $\pi_5^2/Z_{12}=Z_2$. Since ∂_5^2 is trivial, j_5^2 is onto and therefore isomorphism onto, It is easily seen that a generator ν' of $\pi_5^2=Z_{24}$ is represented by the mapping $\tilde{\nu}_{n+2}|S^{n+5}$. Since $\pi_4=0$, we have $C^2(4)=0$ and

Proposition (8.4) $\pi_{n+5}(S^n)=0$ *for* $n\geqq 7$.

iv) The generators of $\pi_5^5=Z_2+Z_2$ are $i_5^5(\eta')$ and $i_5^5\circ i_5^4\circ i_5^3(\nu')$, and they are also the image of ∂_6^6. Hence $\pi_5^6=0$ and K_n^{n+6} has the homotopy groups $\pi_i(K_n^{n+6})=0$[13] for $n<i<n+6$. From (6.4) and (6.1) we have

$$0=H_{n+7}(K_n)=\pi_6^5/image\ \partial_7^6,$$

and ∂_7^6 is onto. An analogeous consideration as in iii) and the fact $2\nu_n\circ\nu_{n+3}=0$ show that

$$\begin{aligned}
&image\ \partial_7^6=Z_2+Z_2 \text{ or } Z_2=\{\eta'\circ\eta_{n+5},\nu'\circ\eta_{n+5}\},\quad image\ j_6^5=0,\\
&image\ \partial_7^5=Z_2=\{\xi_n\circ\eta_{n+4}\circ\eta_{n+5}\}\quad image\ j_6^4=Z_2,\\
&image\ \partial_7^4=Z_2 \text{ or } 0=\{\nu_n\circ\nu_{n+3}\},\quad image\ j_6^3=Z_2,\\
&image\ \partial_7^3=image\ j_6^2=image\ \partial_7^2=0,
\end{aligned}$$

and that there are elements $\alpha_1\in\pi_6^4$ and $\alpha_2\in\pi_6^3$ such that $i_6^6(\alpha_1)=\eta'\circ\eta_{n+5}$, $i_6^4(\alpha_2)$

13) Cf. (4.6).

$=\xi_n \circ \eta_{n+4} \circ \eta_{n+5}$, $j_6^4(\alpha_1) \neq 0$ and $j_6^3(\alpha_2) \neq 0$. Consequently we have

Proposition (8.5) *$\pi_{n+6}(S^n)=0$ or Z_2 or Z_2+Z_2 for $n \geqq 8$.*

v) To prove the non-triviality of π_6, we construct a complex K_n^{n+7} whose homotopy groups $\pi_i(K_n^{n+7})$ vanish for $n<i<n+7$. If we assume $\pi_6=0$, we can prove that the group $H_{n+8}(K_n)=\pi_7^5/image\ \partial_8^7$ must contain a group of 8-element and this contradict to (6.1).

Proposition (8.6) *$\pi_{n+6}(S^n)=Z_2$ or Z_2+Z_2 for $n \geqq 8$, and the generators of which are $\nu_n \circ \nu_{n+3}$ and an element of $\{\eta_n, \nu_{n+1}, \eta_{n+4}\}$.*

Fur ther calculations show

Proposition (8.7) *If $n \geqq 9$ we have $\pi_{n+7}(S^n)=Z_{15}+G$ where G is a group of 2^k elements $(3 \leqq k \leqq 8)$.*

ii) *If $n \geqq 10$, we have that $\pi_{n+8}(S^n)$ is a group of 2^k elements.*

Appendix 1. The homotopy groups of the suspended space of the projective plane.

Let Y^2 be the real projective plane, and let $Y^2=S^1 \cup e^2$ be its cell decomposition in which the cell e^2 is attached to S^1 by a mapping of degree 2. Let Y^{n+1} be the $(n-1)$-fold suspended space of Y^2, then $Y^{n+1}=S^n \cup e^{n+1}$ is also a cell complex with a characteristic mapping $\tilde{\omega}:(E^{n+1}, S^n) \rightarrow (Y^{n+1}, S^n)$ such that $\tilde{\omega} | S^n=\omega$ is a mapping of degree 2. By (1.26) the characteristic mapping $\tilde{\omega}$ induces the isomorphism $\tilde{\omega}^*: \pi_p(E^{n+1}, S^n) \rightarrow \pi_p(Y^{n+1}, S^n)$ for $p \leqq 2n-2$. Since the boundary homomorphism $\partial: \pi_p(E^{n+1}, S^n) \rightarrow \pi_{p-1}(S^n)$ is isomorphism, we obtain an exact sequence $\cdots \xrightarrow{\omega^*} \pi_p(S^n) \xrightarrow{i^*} \pi_p(Y^{n+1}) \xrightarrow{\Delta} \pi_{p-1}(S^n) \xrightarrow{\omega^*} \pi_{p-1}(S^n) \longrightarrow \cdots$ by setting $\Delta=\partial \circ \omega^{*-1} \circ j$ for $p \leqq 2n-2$. Since $E: \pi_{p-2}(S^{n-1}) \rightarrow \pi_{p-1}(S^n)$ is onto for $p \leqq 2n-1$, we have $\omega^*(\alpha)=2\iota_n \circ \alpha=2\iota_n \circ E\alpha'=2E\alpha'=2\alpha$, and therefore the kernel of ω^* is the subgroup ${}_2[\pi_{p-1}(S^n)]$ and the image of i^* is isomorphic to $\pi_p(S^n)/2\pi_p(S^n)$.

Lemma *If γ is an element of $\pi_p(Y^{n+1})$ such that $\Delta(\gamma)=\alpha \in \pi_{p-1}(S^n)$, then $2\gamma=i^*(\alpha \circ \eta_n)$.*

The lemma follows from (5.9) and (5.7). Applying this lemma to the results of $\pi_n(S^r)$ we have

Theorem
i) $\pi_n(Y^{n+1})=Z_2 \quad n \geqq 1$,
ii) $\pi_{n+1}(Y^{n+1})=Z_2 \quad n \geqq 3$,
iii) $\pi_{n+2}(Y^{n+1})=Z_4 \quad n \geqq 4$,
iv) $\pi_{n+3}(Y^{n+1})=Z_2+Z_2 \quad n \geqq 5$,
v) $\pi_{n+4}(Y^{n+1})=Z_2 \quad n \geqq 6$,
vi) $\pi_{n+5}(Y^{n+1})=0 \quad n \geqq 7$,
vii) $\pi_{n+6}(Y^{n+1})=Z_2$ or $Z_2+Z_2 \quad n \geqq 8$.

Appendix 2. Lower dimensional cases

Recently P. Sree, [16] has provided that there is a homomorphism: $\pi_{n-2}(S^{2r-3}) \to \pi_n(S^r\,; E^4_+, E^r_-)$ and which is onto for $n \leqq 3r-4$ and isomorphic for $n<3r-4$. By (3.13) $\pi_n(S^r\,; E^r_+, E^r_-)$ has a direct factor isomorphic to $\pi_{n+1}(S^{2r})$ for $n \leqq 3r-4$. Therefore the homomorphism $P: \pi_{n-r+1}(E^r_-, S^{r-1}) \to \pi_n(S^r\,; E^r_+, E^r_-)$ given by $P(\alpha)=[\alpha, \iota_r]_t$ is isomorphism onto for $n \leqq 3r-4$, where ι_r is a generator of $\pi_r(E^r_+, S^{r-1})$. The exctness of the sequence $\pi_{n+2}(S^{r+1}\,; E^{r+1}_+, E^{r+1}_-) \xrightarrow{\Delta} \pi_n(S^r) \xrightarrow{E} \pi_{n+1}(S^{r+1})$ shows that the kernel of suspension homomorphism $E: \pi_n(S^s) \to \pi_{n+1}(S^{r+1})$ is generated by the Whitehead product $[\alpha, \iota_r]$ $(\alpha \in \pi_{n-r+1}(S^r))$ for $n \leqq 3r-3$.

The following list of special Whitehead product is verified $(n \leqq 3r-3)$

n	8	9	9	10	11	11	12	13
r	4	5	4	5	6	5	6	7
	$[\eta_4, \iota_4] \neq 0$	$[\iota_5, \iota_5] \neq 0$	$[\eta_4 \circ \eta_5, \iota_4] \neq 0$	$[\eta_5, \iota_5] \neq 0$	$[\iota_6, \iota_6] \neq 0$	$[\eta_5 \circ \eta_6, \iota_5] \neq 0$	$[\eta_6, \iota_6] \neq 0$	$[\iota_7, \iota_7] \neq 0$

$[\eta_6, \iota_6]=0$ and $[\iota_7, \iota_7]=0$ follow from (2.26) and the fact that there are mappings of types (η_6, ι_6) and (ι_7, ι_7). By (7.3), (2.23) and (4.6) we have $[\eta_5 \circ \eta_6, \iota_5] = \nu_5 \circ \eta_8 \circ \eta_9 \circ \eta_{10} = \nu_5 \circ 12\nu_8 = 12\nu_5 \circ \nu_8 = 0$. Since $H[\iota_6, \iota_6]=2\iota_{12} \neq 0$ we have $[\iota_6, \iota_6] \neq 0$. The fact $[\eta_4, \iota_4]=\alpha_4 \circ \eta_7 \neq 0$ and $[\iota_5, \iota_5]=\nu_5 \circ \eta_8 \neq 0$ is already verified in (4.4) and (4.5). From (2.23) we have $[\eta_4 \circ \eta_5, \iota_4]=\alpha_4 \circ \eta_7 \circ \eta_8$ and $[\eta_5, \iota_5]=\nu_5 \circ \eta_8 \circ \eta_7$. Since $H(\alpha_3 \circ \eta_6 \circ \eta_7)=\eta_6 \circ \eta_7 \circ \eta_8 \neq 0$ and $H(\nu_4 \circ \eta_7 \circ \eta_8)=\eta_8 \circ \eta_9 \neq 0$, we have by (4.2)′ $E(\alpha_3 \circ \eta_6 \circ \eta_7) \neq 0$ and $E(\nu_4 \circ \eta_7 \circ \eta_8) \neq 0$.

Therefore the exactness of the sequence $\cdots \pi_{n+1}(S^r) \xrightarrow{E} \pi_{n+2}(S^{r+1}) \xrightarrow{I} \pi_{n+2}(S^{r+1}\,; E^{r+1}_+, E^{r+1}_-) \xrightarrow{\Delta} \pi_n(S^r) \xrightarrow{E} \pi_{n+1}(S^{r+1}) \to \cdots$ leads the following reselts;

i) $E: \pi_8(S^4) \to \pi_9(S^5)$ *is onto and its kernel is generated by* $\eta_4 \circ \nu_5$,

ii) $E: \pi_9(S^5) \to \pi_{10}(S^6)$ *is onto and its kernel is generated by* $\nu_5 \circ \eta_8$,

iii) $E: \pi_{10}(S^6) \to \pi_{11}(S^7)$ *is onto its kernel is generated by* $\eta_4 \circ \nu_5 \circ \eta_8$,

iv) $E: \pi_{10}(S^5) \to \pi_{11}(S^6)$ *maps into the subgroup of* $\pi_{11}(S^6)$ *which is generated by the elements of the Hopf invariants* 0, *and the kernel of* E *is generated by* $\nu_5 \circ \eta_8 \circ \eta_9$.

v) $E: \pi_{11}(S^6) \to \pi_{12}(S^7)$ *is onto and its kernel is generated by* $[\iota_6, \iota_6]$,

vi) $E: \pi_{11}(S^5) \to \pi_{12}(S^6)$ *is isomorphism onto,*

vii) $E: \pi_{12}(S^6) \to \pi_{13}(S^7)$ *is isomorphism onto,*

viii) $E: \pi_{13}(S^7) \to \pi_{14}(S^8)$ *is isomorphism onto.*

Summarizing the results of $\pi_n(S^r)$ we obtain;

a) $\pi_n(S^n)=Z$ *for* $n \geqq 1$, $\pi_n(S^1)=0$ *for* $n>1$ *and* $\pi_n(S^r)=0$ *for* $n<r$.

b) $\pi_3(S^2)=Z=\{\eta_2\}$ *and* $\pi_{n+1}(S^n)=Z_2=\{\eta_n\}$ *for* $n \geqq 3$,

c) $\pi_{n+2}(S^n)=Z_2=\{\eta_n\circ\eta_{n+1}\}$ *for* $n\geqq 2$,

d) $\pi_5(S^2)=Z_2=\{\eta_2\circ\eta_3\circ\eta_4\}$, $\pi_6(S^3)=Z_{12}=\{\alpha_3\}$, $\pi_7(S^4)=Z+Z_{12}=\{\nu_4\}+\{\alpha_4\}$ *and* $\pi_{n+3}(S^n)=Z_{24}=\{\nu_n\}$ *for* $n\geqq 5$,

e) $\pi_6(S^2)=Z_2=\{\eta_2\circ\alpha_3\}$, $\pi_7(S^3)=Z_2=\{\eta_3\circ\nu_4\}$, $\pi_8(S^4)=Z_2+Z_2=\{\eta_4\circ\nu_5\}+\{\nu_4\circ\eta_7\}$, $\pi_9(S^5)=Z_2=\{\nu_5\circ\eta_8\}$ *and* $\pi_{n+4}(S^n)=$ *for* $n\geqq 6$.

f) $\pi_7(S^2)=Z_2=\{\eta_2\circ\eta_3\circ\nu_4\}$, $\pi_8(S^3)=Z_2=\{\eta_3\circ\nu_4\circ\eta_7\}$, $\pi_9(S^4)=Z_2+Z_2=\{\eta_4\circ\nu_5\circ\eta_8\}+\{\nu_4\circ\eta_7\circ\eta_8\}$, $\pi_{10}(S^5)=Z_2=\{\nu_5\circ\eta_8\circ\eta_9\}$, $\pi_{11}(S^6)=Z=[\iota_6,\iota_6]$ *and* $\pi_{n+5}(S^n)=0$ *for* $n\geqq 7$.

g) $\pi_{n+6}(S^n)=Z_2=\{\nu_n\circ\nu_{n+3}\}$ *or* $=Z_2+Z_2=\{\nu_n\circ\nu_{n+3}\}+\{\eta_n,\nu_{n+1},\eta_{n+4}\}$ *for* $n\geqq 5$.
The essentiality of $\nu_n\circ\nu_{n+3}$ is follows from (4.2).

References

[1] M. G. Barratt and G. F. Paecher, *A note on* $\pi_r(V_{n,m})$, Pro. Nat. Acad. Sci. U. S. A. 38 (1952), 119-121.

[2] A. L. Blakers and W. S. Massey, *The homotopy groups of a triad I*, Ann. of Math. 53 (1951), 161-205.

[3] A. L. Blakers and W. S. Massey, *The homotopy groups of a triad II*, Ann. of Math. 55 (1952), 192-201.

[4] H. Carten and J. P. Serre, *Espace fibrés et groupes d'homotopie I, II.* C. R. 234 (1952), 288-290, 393-395.

[5] B. Eckmann. *Zur Homotopietheorie gefaserter Räume*, Comm. Math. Helv. 14 (1941), 234-256.

[6] S. Eilenberg and S. MacLane, *Relations between homology and homotopy groups of spaces*, Ann. of Math. 46 (1945), 480-509.

[7] S. Eilenberg and S. MacLane, *Cohomology theory of abelian groups and homotopy theory I, II, III*, Pro. Nat. Acad. Sci. U. S. A. 36 (1950), 443-447, 657-663, 37 (1951) 307-310.

[8] H. Freudenthal, *Über die Klasse der Sphären abbildungen*, Comp. Math 5 (1937) 299-314.

[9] P. Hilton, *Suspension theorems and the generalized Hopf invariant*, Proc. London Math. Soc. 1 (1951), 463-493.

[10] H. Hopf, *Über die Abbildungen der dreidmensionalen Sphären auf die Kugelfläche*, Math, Ann. 104 (1931), 637-665.

[11] H. Hopf, *Über die Abbildungen von Sphären auf Sphären niedrigerer Dimension*, Fund. Math. 25 (1935), 427-440.

[12] W. Hurewicz and N. E. Steenrod, *Homotopy relations in fibre spaces*, Proc. Nat. Acad. Sci. U. S. A. 27 (1941), 60-64.

[13] T. Kudo, *Homological structure of fibre bundles*, Jour. Inst. Poly, Osaka City Univ. 2-2 (1952), 101-140.

[14] J. P. Serre, *Homologie singulière des espaces fibrés*, Ann. of Math. 54 (1951) 425-505.

[15] J. P. Serre, *Sur les groupes d'Eilenberg-MacLane*, C. R. 234 (1952) 1243-1245.

[16] J. P. Serre, *Sur la suspension de Freudenthal*, C. R. 234 (1952), 1340-1342.

[17] N. E. Steenrod, *Theory of fibre bundle*, Princeton Univ. Press.

[18] H. Toda, *Some relations in homotopy groups of spheres*, Jour. Inst. Poly. Osaka City Univ. 2-2 (1952) 71-80.

[19] G. W. Whitehead, *Homotopy properties of the real orthogonal groups*, Ann, of Math. 43 (1942), 132-146.

[20] G. W. Whitehead, *On the homotopy groups of spheres and rotation groups*, Ann. of Math. 43 (1942), 634-640.

[21] G. W. Whitehead, *On products in homotopy groups*, Ann. of Math. 47 (1946), 460-475.
[22] G. W. Whitehead, *A generalization of the Hopf invariant*, Ann. of Math. 51 (1950), 192-237.
[23] G. G. Whitehead, *The* $(n+2)^{nd}$ *homotopy group of the n-sphere*, Ann. of Math. 52 (1950), 245-247.
[24] J. H. C. Whitegead, *On adding relations to homotopy groups*, Ann. of Math. 42 (1941), 409-428.